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Assortative matching through signals*

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Abstract

We model signalling in two-sided sequential search with heterogeneous agents and transferable utility. Search via meetings is time-consuming and thereby costly due to discounting. Search via signals is costless, so that agents can avoid almost all search costs if only the signals are truthful. We show that signals will indeed be truthful if the match output function is sufficiently supermodular. The unique separating equilibrium is then characterised by perfect positive assortative matching despite the search frictions. In this equilibrium, agents successfully conclude their search after a single meeting, and overall match output is maximised. These results continue to hold when there are also explicit search costs in addition to discounting.

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1 Introduction

A number of important markets bring trade partners together in pairs. In such two-sided matching markets, it is a pervasive phenomenon that likes tend to match with likes: more accomplished workers tend to be hired by more successful firms, more educated women tend to marry more educated men, more reliable tenants tend to secure nicer apartments.[1] This sorting of likes along some dimensions, known as *positive assortative matching* (PAM), is relevant to economic research in at least two ways. First, assortative matching is typically more efficient than, say, random matching: highly skilled workers are better placed in complex work environments than unskilled workers, for example. If economists understand what drives assortative matching, they will be in a position to avoid mismatch in market design or to improve efficiency in existing markets by making PAM more pronounced. Second, PAM serves as a test on the validity of theoretical models: any model of a matching market with heterogeneous agents that cannot generate PAM appears to have missed something important.

There is a small and young literature focussing on how assortative matching is generated. The literature identifies some form or another of complementarity among the inputs into a match as the driver of PAM. Thinking of the value generated by a match as the match output, a match production function specifies how agents' inputs translate into match output. Then simple complementarity among the inputs is equivalent to *supermodularity* of the match production function: the marginal effect on output from a change in one input is increasing in the other input. However, simple complementarity among inputs, i.e. supermodularity as such, often does not suffice for PAM to arise, and various more complicated forms of supermodularity are then resorted to.

Following Smith (2006), we would classify the models in the literature by two criteria. One is whether or not utility is transferable between agents. The literature refers to *non-transferable utility* whenever agents divide the match output according to a pre-imposed split, like an employer and her employee do when the employee's wage is already determined by an agreement between unions and employer associations.[2] In cases of *transferable utility*, there is no pre-imposed split and agents have to bargain over the match output and agree on a split. The other criterion is the kind of frictions in the model as captured by the costs of search. The seminal article by Becker (1973) considers a frictionless setting (i.e. agents search costlessly) where utility is transferable, and supermodularity as such turns out to suffice for PAM, in fact even perfect PAM: the types on one side of the market match exclusively with exactly corresponding types on the other side.[3] In a setting with frictions, the costs of search may be expressed as implicit costs through discounting or as explicit additive costs. An influential contribution by Shimer and Smith (2000) examines a setting with discounting and transferable utility. They conclude that the match production function, the logarithm of its first derivative, and the logarithm of its cross-partial derivative all need to be supermodular for PAM to arise in this setting. For the same setting with non-transferable utility, Smith (2006) shows that PAM will only arise if the match production function is log-supermodular.

While supermodularity as such is a very natural condition, the combination of three conditions required for PAM in Shimer and Smith (2000) is criticised by Atakan (2006)

[1] As an exemplary reference for these stylised facts, see Mare (1991).

[2] See Burdett and Coles (1999), p. F311.

[3] Using Becker's (1973) work, it is quickly found that a frictionless setting with non-transferable utility leads to PAM even without supermodularity: see Smith (2006), section II.

as restrictive.[4] His own model features transferable utility, as in Shimer and Smith (2000), but explicit costs instead of discounting. Morgan (1998) builds a model with non-transferable utility and explicit costs.[5] In both models, supermodularity as such gives rise to PAM. However, limiting oneself to explicit costs provides no answer to the paradox that real-world agents in all sorts of matching markets both discount and sort into PAM despite the restrictive conditions identified by Shimer and Smith (2000). Nor is it a solution to conclude that utility must then be non-transferable in most real-world matching markets - many if not most real-world matching markets involve some form of bargaining, so that utility must be transferable. And even if utility was typically non-transferable, production functions would still have to be log-supermodular according to Smith (2006), arguably not the least restrictive of conditions.

This paper offers a solution by appealing to another, equally pervasive phenomenon in matching markets: the use of signals. Signals come in the form of job advertisements and applications on the labour market, dress style and body language on the marriage market, online photographs of apartments and flat hunters on the market for rented housing, to name but a few. Such signals may allow agents to search selectively, thereby avoiding some explicit search costs, and to conclude search sooner, thereby limiting implicit search costs overall. Hence, signals can potentially reduce the effect of search frictions and thus bring a setting with frictions closer to a frictionless setting, in which mild conditions suffice for PAM to arise. Of course, signals will only have this beneficial effect if they are actually informative. In the famous signalling model by Spence (1973), years of education are an informative signal for ability because more able workers find it easier to acquire education than less able workers do. The model thus relies on signals being costly and on a single-crossing property. Unfortunately, when agents signal their type through applications or advertisements, the costs are typically small and a single-crossing property can in general not be expected to hold: writing a forged CV is as costly as writing a truthful CV, and painting an advertised job in unduly bright colours is as costly as honestly laying out its dull nature. Yet, as shown by contributions such as Crawford and Sobel (1982), signals will be informative even in an environment of such *cheap talk* if the interests of senders and receivers are sufficiently aligned. Then the costs of signals can just as well be normalised to zero.

The model we build introduces costless signals into a search model with transferable utility very close to that of Shimer and Smith (2000). Our model is more general in so far as explicit search costs are also included and as different parameters apply to the two sides of the market, while parameters are identical and explicit costs are zero in Shimer and Smith (2000). Throughout the paper, we spell out our key results also for this special case, while we would in general argue that real-world differences between the two sides of the market justify differential treatment. Our model is less general in so far as we assume discrete uniform type distributions instead of general type distributions. We prove a unique separating equilibrium in which matching is not only positively assortative but perfectly positively assortative whenever the match production function is sufficiently supermodular (and explicit search costs are not prohibitively high). In this equilibrium, each agent finds it optimal to signal truthfully and to target her search on only one type. The key idea is that an agent will in fact not necessarily prefer matches with higher types because higher types appropriate larger shares of match output, thanks to more

[4] See Atakan (2006), p. 667.

[5] Morgan (1998) is the revised version of the mimeo with the same title dating from 1995.

comfortable fall-back positions in bargaining. Indeed, sufficient supermodularity leads to a preference for an exactly corresponding type, thereby perfectly aligning the interests of agents from the two sides of the market and ensuring informative signals. Intuitively, perfect PAM is then to be expected: if signals are fully informative, agents can replace (almost all) costly search via meetings by costless search via signals and can therefore behave like in a frictionless setting.

The separating equilibrium has a number of desirable efficiency properties. Above all, agents match at the very first opportunity, so that no time is wasted on unsuccessful search and search costs are minimised. In labour market terms, this would mean that frictional unemployment disappears almost completely. An extension offers a realistic scenario implying that a weaker condition on the supermodularity of the match production function can suffice to ensure truthful signalling in reality. In any case, it appears that our model requires less restrictive conditions than the model by Shimer and Smith (2000), and yet it achieves perfect PAM instead of merely PAM. Finally, we report non-separating equilibria of our model, which are a special case of the equilibrium in Shimer and Smith (2000), except for elements specific to models with signals.

Our paper belongs to a small wave of current papers that consider signalling in the context of matching. In a model built by Chade (2006), a noisy signal uncontrolled by the agent provides some information about the agent's type. Although agents discount and no supermodularity is assumed (albeit some other conditions), matching is shown to exhibit PAM in a stochastic sense: the distribution of types a high type might match with first-order stochastically dominates this distribution for a low type. Chade (2006) further finds that an assumption of log-supermodularity of the production function reinforces these results. Our set-up primarily differs from Chade's (2006) in that signals are not stochastic but deliberately chosen by agents. We believe real-world agents will exert as much control as possible over the signals of their types, given how important the signals can be for their payoffs. A comparison between Chade's (2006) model and ours thus gives an indication of the efficiency gains from all agents learning to control their signals.

Next, Hopkins (2007) and Hoppe et al. (2008) build two similar models of a *matching tournament* with signalling: match partners are essentially prizes for ex-ante choices of costly signals. In both models, agents first select a costly signal of their unobservable type like in Spence (1973) and then, based on these signals, match roughly like in Becker (1973). In the symmetric equilibrium, agents' signals are strictly increasing in their types, even without the assumption of a single-crossing property. Assuming supermodularity whenever utility is transferable (otherwise, no assumption is needed), this then leads to perfect PAM at the matching stage - just as one would have expected, given Becker's (1973) findings. That a single condition on supermodularity ensures both informative signals and perfect PAM in Hopkins (2007) and Hoppe et al. (2008) is a parallel to our model: no additional assumption of a single-crossing property or, in our cheap-talk setting, the sufficient alignment of agents' interests is needed. However, since search frictions do not exist in matching tournaments, neither of the two papers helps us to resolve the paradox in Shimer and Smith (2000).

Finally, a model by Eeckhout and Kircher (2008) features the key elements of models of *directed search* (also known as competitive search), including signals in the form of sellers' posted offers. Buyers observe the posted offers and simultaneously choose which seller to visit in the current round. The only way frictions enter this process is through congestion: buyers cannot coordinate, so that queues result and only some buyers manage to buy in each round. It is shown that the matching of buyers and sellers will exhibit PAM if the square root of the match production function is supermodular.[6] Shimer (2005) looks at PAM in a standard directed search model of the labour market and finds, at least for the case of only two worker types, that there will be some stochastic form of PAM as long as workers of low type do not have a comparative advantage when working for employers of high type. While these models simply assume that signals are truthful (a criticism we would level against almost any directed search model), these models cannot resolve the paradox mainly because of their limitation to frictions from congestion. Findings on signals in models with these frictions unfortunately do not carry over to models with other frictions in any obvious way.

Apart from the differences to existing models we have outlined here, three unique features of our model are worth noting. To the best of our knowledge, it is the only model that generates perfect PAM despite discounting. Moreover, it appears to be the only model to examine assortative matching with both discounting and explicit search costs, and consequently the only model to generate perfect PAM even in this context. Nevertheless, we have the impression that our model is considerably simpler than any model of signalling in a matching context that we are aware of.

The paper proceeds as follows. Section 2 specifies the environment and procedures of search in our model. Section 3 characterises a putative separating equilibrium whose main parts are to be examined one at a time. Section 4 derives the search strategies from optimal behaviour. Section 5 identifies a condition on supermodularity under which signalling will be truthful. In section 6, the proof of perfect PAM under this same condition completes the proof of the separating equilibrium, which is concisely stated and found to be unique as well as efficient in section 7. Section 8 extends by deriving a weaker condition on supermodularity from a different but realistic scenario. Section 9 considers non-separating equilibria, section 10 explores the relation of our model to directed search models, and section 11 concludes.

2 Set-up

2.1 Search environment

The two sides of the matching market in our model are a set N_X of agents indexed by a productivity type $x \in \Theta$ and a set N_Y of agents likewise indexed by $y \in \Theta$, where Θ is a large but finite subset of $[0, 1]$. The numbers of agents, $|N_X|$ and $|N_Y|$, are both finite but sufficiently large so that each agent's market power ex ante is negligible. The productivity types are exogenously given and verifiable, but not distantly observable. To convey information about their types from a distance, agents can freely choose any element of Θ as a public and costless signal of their type. Types are discretely distributed according to the respective distribution functions $L_X : N_X \mapsto \Theta$ and $L_Y : N_Y \mapsto \Theta$ that have the same discrete support, which may be justified by thinking of types as ranks in a ranking over agents on the respective side of the market.

Assumption 1 (Type distributions). *In steady state, the distributions L_X and L_Y are uniform with probability mass functions l_X and l_Y, respectively.*

[6] A somewhat comparable model is offered by Coles and Niederle (2007) where buyers signal their preferences over sellers to the sellers, who then determine which buyer to offer the transaction. However, the issue of PAM is not considered in their paper.

Assumption 1 allows us to rule out that any part of agents' behaviour is driven by differences in the frequency of types. To this end, it might well suffice to assume a smooth type distribution, so that there are no dramatic differences in the probabilities of adjacent types. By specifying uniform distributions we simply select perfectly smooth distributions. In fact, when there are dramatic discontinuities between adjacent types' probabilities, this will typically create incentives for agents close to this discontinuity to slightly change their type. The standard logic of arbitrage thus suggests that discontinuities will be evened out at least in steady state, so that assumption 1 does not seem totally implausible. While it is common in the real world for secretaries to become call centre agents when there is an abundance of secretaries and a shortage of call centre agents, we do not attempt to model such type changes here.

Under assumption 1, the identical discrete support of $L_X(\cdot)$ and $L_Y(\cdot)$ conveniently implies an equal number of different types on both sides of the market: $L_X(\cdot)$ and $L_Y(\cdot)$ span an identical support without gaps. Normalising the flow output generated by an unmatched agent to zero, a match between a type x and a type y generates a constant flow output $f(x, y)$.

Assumption 2 (Regularity conditions). *The match production function $f(\cdot, \cdot)$ is symmetric ($f(x, y) \equiv f(y, x)$) and takes only positive values ($f : \Theta^2 \mapsto \mathbb{R}^+$).*

Our assumption of symmetry follows the literature. It may again appear less restrictive if one thinks of inputs x and y as ranks, which are typically much more comparable than concrete productive inputs and can therefore reasonably be assumed to enter the production function in the same way.

Assumption 3 (Supermodularity). *For all $x > x'$ and $y > y'$, the match production function satisfies $f(x, y) - f(x', y) > \gamma\,[f(x, y') - f(x', y')]$ where $\gamma \geq 1$. We call this "strict supermodularity of degree γ".*

The standard notion of supermodularity would have $\gamma = 1$ and would mean that the effect on output from a change in one input is increasing in the other input. The notion here requires that this increase be strong enough. One aim of our analysis will be to identify which value of γ is needed in equilibrium. Unfortunately, our notion of supermodularity relies on sufficiently discrete type distributions: for any given $\gamma > 1$, the statement in assumption 3 will eventually fail if y' is allowed to become arbitrarily close to y. (As the definition of a derivative involves taking the limit such that $(y - y') \to 0$, assumption 3 therefore cannot be expressed in terms of a cross-partial derivative.) The minimum distance between two types in L_Y so that assumption 3 just holds can be the smaller the smaller this distance is in L_X and vice versa.[7] The minimum distance must be the greater the greater is γ.

Since utility is transferable, the parties to the match have to bargain over the division of the match output before the match is consummated. Each agent's flow utility during the match equals the output share obtained. An agent x discounts future utility at discount rate r_X, while an agent y discounts at rate r_Y, with $0 < r_X, r_Y < \infty$. All agents are

[7] To see this, let y' become closer and closer to a given y. Because of supermodularity, the difference $f(x, y') - f(x', y')$ will grow, and will grow the faster the more different x and x' are. Hence, with more different x and x', a growing y' will sooner run into the constraint that assumption 3 has to hold, thus leaving a greater difference between y and y'. In turn, let x' become closer and closer to a given x. Because of supermodularity, $f(x, y) - f(x', y)$ will fall faster than $f(x, y') - f(x', y')$, and the more so the more different y and y' are.

risk-neutral and seek to maximise the present discounted value (pdv) of their expected utility. Throughout the paper, 'payoff' refers to the pdv, not to the flow utility.

2.2 Search procedure

Our model features two-sided search in continuous time with an infinite horizon. Consider an agent x who enters the pool of searchers at some point in time. Upon entering, she begins sending a costless and public signal denoted $\tilde{x} \in \Theta$, which may or may not be an accurate signal of her type x. She can always instantly and costlessly change the signal she is sending. In turn, she observes sequentially the signals sent by the other side of the market, while she cannot observe the true types at this stage. Whenever she observes such a signal $\tilde{y} \in \Theta$, she can only form a belief $\mu_X(y|\tilde{y})$ about the probability that the true type is y, given the signal $\tilde{y}$. The time and the costs involved in observing and evaluating signals are negligible. What we have in mind here is best represented by an online platform where the next advertisement is only a click away.

However, before x can match with some agent y, a meeting between the two will have to occur. Naturally, a meeting between two agents will only occur if each is willing to meet the other, given the observed signals. Precisely because agents observe signals, meetings are non-random: agents *arrange* meetings only with agents they would like to meet. Thanks to modern technology, communication between agents only takes a negligible amount of time. Once two agents have agreed to meet, however, the meeting does not take place instantly. Rather, opportunities for meetings only present themselves at certain rates.

Assumption 4 (Arrival rates). *Any agent $x \in \Theta$ manages to meet an agent y to whom she appears acceptable at exogenous Poisson rate η_X. Likewise, any agent $y \in \Theta$ manages to meet an agent x to whom she appears acceptable at exogenous Poisson rate η_Y, with $0 < \eta_X, \eta_Y < \infty$.*

While agents on the same side of the market may face slightly different meeting rates, the point of assumption 4 is that any such differences do not affect search behaviour, so that all agents on one side of the market behave as if they faced identical meeting rates. Of course, assumption 4 is strong nevertheless. We make this assumption because we do not see causes for substantively different meeting rates in our model: given assumption 1, different relative frequencies of types cannot be a cause here, nor can any notion of distance, since communication through modern technology is equally rapid for all types. Among the few potential causes that remain are congestion frictions, but section 10 will explain why congestion frictions do not arise in our model and why our set-up actually all but implies identical meeting rates.

What we would argue underlies agents' inability to meet instantly are practical issues that apply to all types more or less equally: agents might be busy with other things, so that they only have time for a meeting at some later point, or agents need time to travel to the meeting, or they might deliberately allow some time to prepare for this meeting. All agents are therefore constrained by how many arranged meetings they can attend over a given time period. Because of discounting, the time that elapses before a meeting makes meetings costly, as opposed to observing and evaluating signals. In addition, any agent $x \in \Theta$ incurs explicit costs $c_X \geq 0$ each time she attends a meeting, while any agent $y \in \Theta$ incurs $c_Y \geq 0$. Note that a model without explicit search costs will result if c_X and c_Y are set to zero.

During the meeting, the agents bargain over the division of the match output that they would produce between them.[8] The shares agreed are given by the *generalised Nash bargaining solution* where $q \in (0,1)$ represents the bargaining power of any agent $x \in \Theta$ and $1-q$ that of any agent $y \in \Theta$.[9] Since agents cannot recall previous offers, they can only threaten to walk out of the meeting and continue searching. If both agents agree to match, they each receive their agreed share of $f(x,y)$ as flow utility. For the duration of a match, shares are constant and the agents involved do not belong to the pool of searchers (i.e. on-the-job search is ignored). Matches dissolve exogenously at constant Poisson rate δ, and in this case the agents return to the pool of searchers. In all this, we assume agents to have the following information:

Assumption 5 (Information). *Each agent knows her own type, the match production function $f(\cdot,\cdot)$, discount rates r_X and r_Y, costs c_X and c_Y, the parameter of bargaining power q as well as the rates η_X, η_Y, and δ.*

Before moving on, we finally define a number of sets that will be useful in the analysis below. Let $D(x)$ denote the set of signals $\tilde{y}$ such that an agent of type x would like to meet the agent sending such a signal, that is, the *meeting strategy* (or, in marriage market terms, the dating strategy) of agent x. Then the set of signals $\tilde{y}$ whose senders would be willing to meet x, given her signal $\tilde{x}$, is her *opportunity set*

$$\Omega(x) \equiv \{\tilde{y} : \tilde{x} \in D(y)\}.$$

Combining these sets, $R(x) \equiv D(x) \cap \Omega(x)$ denotes the set of signals $\tilde{y}$ such that a meeting (or a *rendez-vous*) between x and y would in fact result. Next, the set of types y whom agent x would accept for a match constitutes her *stopping rule* $S(x)$. Then the *matching set* is the set of types y such that a match between x and y would result:

$$M(x) \equiv S(x) \cap \{y : x \in S(y)\}.$$

Of course, $D(y)$, $\Omega(y)$, $R(y)$, $S(y)$, and $M(y)$ are all defined analogously.

2.3 Steady state

We denote the probability mass function for unmatched agents by $u_X(\cdot) \leq l_X$ and $u_Y(\cdot) \leq l_Y$, respectively. Then $u_X(x)|N_X|$ gives the mass of unmatched agents of type x, while $[l_X - u_X(x)]|N_X|$ gives the mass of matched agents of type x. These quantities and the corresponding quantities for agents with a type $y \in \Theta$ are determined endogenously. Previously matched agents of type x flow into the pool of searchers at rate δ, while previously unmatched agents of type x flow out of the pool at rate η_X times the probability that the meeting leads to a match. In a *pointwise steady state*, inflow and outflow just balance for every type:

$$\delta[l_X - u_X(x)]|N_X| = \eta_X \Pr(y \in M(x)|\tilde{y} \in R(x)) u_X(x)|N_X| \tag{1}$$

for all $x \in \Theta$ and correspondingly

$$\delta[l_Y - u_Y(y)]|N_Y| = \eta_Y \Pr(x \in M(y)|\tilde{x} \in R(y)) u_Y(y)|N_Y| \tag{2}$$

for all $y \in \Theta$. As agents only match in pairs, the total flow of agents $x \in \Theta$ out of (or into) the pool of searchers must always equal the total flow of agents $y \in \Theta$.

[8]The fact that there is no uncertainty about the (future) match output is termed a *no learning restriction* by Burdett and Coles (1999) (p. F311).

[9]We limit ourselves to $q \in (0,1)$ rather than $q \in [0,1]$ here because the generalised Nash bargaining solution would not be defined for $q \in \{0,1\}$.

Example 1 The set-up employed by Shimer and Smith (2000) as well as by Smith (2006) features only one type distribution of agents who match randomly among themselves. Such a set-up is easily derived from our two-sided set-up by letting all differences between the two sides disappear so that only one set of agents exists, indexed by a type $z \in \Theta$. In particular, this requires

$$l_Z \equiv \frac{l_X|N_X| + l_Y|N_Y|}{|N_X| + |N_Y|} \quad \text{and} \quad u_Z(z) \equiv \frac{u_X(x)|N_X| + u_Y(y)|N_Y|}{|N_X| + |N_Y|} \quad \forall x = y$$

as well as $\eta_X = \eta_Y \equiv \eta_Z$ and (normalising) $|N_X| + |N_Y| = 1$. Since identical agents use the same strategies and have the same opportunity sets, it holds here that

$$\Pr(y \in M(x)|\tilde{y} \in R(x)) = \Pr(x \in M(y)|\tilde{x} \in R(y)) \quad \forall x = y.$$

Equations (1) and (2) then add up to the condition for the pointwise steady state:

$$\delta[l_Z - u_Z(z)] = \eta_Z \Pr(z' \in M(z)|\tilde{z}' \in R(z))u_Z(z) \quad \forall z \in \Theta \tag{3}$$

This is the same as the set-up in Shimer and Smith (2000) but for our assumption 1.

3 Putative equilibrium situation

3.1 Concept and present values

As is natural when signals are involved, we look for a *perfect Bayesian equilibrium* (PBE) of our model. We focus our attention on separating equilibria and only turn to equilibria where signals are effectively uninformative in section 9. As signals are costless, all PBE will necessarily be cheap-talk equilibria. We first describe a putative equilibrium situation below that consists of three main building blocks: truthful signalling, individual strategies for meeting and stopping, and *perfect positive assortative matching* (PPAM) in the market; that is, $x = y$ in all matches. We then proceed to prove that these building blocks form indeed a PBE by proving one block at a time, taking the two others as given.

Following this procedure, we will show in section 4 that optimising agents choose their strategies for meeting and stopping as specified in the putative equilibrium situation, given truthful signalling and PPAM. Section 5 will confirm that agents find it optimal to signal truthfully, given their meeting strategies and stopping rules and given PPAM. Finally, section 6 will prove that PPAM results when signals are truthful and agents pursue optimal meeting strategies and stopping rules. A concise characterisation of the separating PBE thus established is then provided in section 7.1. For now, we can only define in general terms (that also apply to non-separating equilibria) what counts as an equilibrium of our model:

Definition 1 (Search equilibrium). *Equilibrium requires that the pointwise steady state holds and that each agent chooses her signal optimally, pursues an optimal meeting strategy, employs an optimal stopping rule, and holds beliefs that are reinforced by equilibrium play.*

The putative equilibrium situation we propose is characterised by the following behaviour. All agents participate in the market, signal their type truthfully, and correctly believe all other agents to be also signalling truthfully. Each agent seeks to meet exclusively those agents in her opportunity set who send the most promising signals; that is, whose signals suggest the type the agent most prefers among the types she can meet. Provided the degree of supermodularity is high enough, each agent most prefers agents of exactly corresponding types, so that meetings only occur between exactly corresponding types: $x = y$ in every meeting. Stopping rules are such that every meeting results in a match, and the matching thus exhibits PPAM.

We now derive the present values that various states carry in the putative equilibrium situation. Let $V(\tilde{x}|\tilde{y})$ be the present value to agent x when she is unmatched while she signals her type as $\tilde{x}$ and would only like to meet an agent signalling $\tilde{y}$ (according to her optimal meeting strategy). Next, let $W(x|y)$ be the present value to x from a match with y. Of course, $V(\tilde{y}|\tilde{x})$ and $W(y|x)$ are defined analogously. As signals are truthful here, $\tilde{x} = x$ and $\tilde{y} = y$. Note that no optimising agent would knowingly attend a meeting that will not lead to a match, due to the costs associated with meetings. When truthful signals allow agents to know everything in advance, every meeting that occurs will therefore lead to a match. Hence an agent x effectively incurs the explicit search costs each time she matches, so that the payoff from matching is $W(x|y) - c_X$. The relation between $V(\tilde{x}|\tilde{y})$ and $W(x|y)$ can be derived from an *asset equation* that equates the return on being unmatched with the gain from being matched incurred at rate η_X:

$$r_X V(\tilde{x}|\tilde{y}) = \eta_X \left[W(x|y) - c_X - V(\tilde{x}|\tilde{y})\right] \tag{4}$$

We can likewise find an asset equation for $W(x|y)$. Recall that the flow utility to an agent from a match equals her share of the match output. For an agent x in a match with an agent y, we denote the share as $\pi(x|y)$. Writing $U(x)$ for the present value of having to begin searching from scratch,

$$r_X W(x|y) = \pi(x|y) - \delta[W(x|y) - U(x)] \tag{5}$$

captures that agent x receives the flow $\pi(x|y)$ while matched but incurs a loss $W(x|y) - U(x)$ if the match dissolves, which happens at rate δ. To close the circle, note that the stationarity of an equilibrium situation implies that x will continue to signal truthfully and seek a meeting with an exactly corresponding type again:

$$U(x) = V(\tilde{x}|\tilde{y}) \quad \text{where} \quad \tilde{x} = x = \tilde{y}. \tag{6}$$

3.2 Shares in bargaining

On the basis of the relations between agents' present values, we now turn to bargaining over the match output $f(x,y)$ (a flow). Because agents will not leave anything on the table, we know that

$$\pi(x|y) + \pi(y|x) = f(x,y) \tag{7}$$

If bargaining fails, each agent simply continues to search. In flow terms, the threat point of agent x is therefore $r_X V(\tilde{x}|\tilde{y})$. By equation (4), this is

$$\begin{aligned} r_X V(\tilde{x}|\tilde{y}) &= \frac{\eta_X}{r_X + \eta_X}\left[r_X W(x|y) - r_X c_X\right] \\ \Leftrightarrow (r_X + \eta_X) r_X V(\tilde{x}|\tilde{y}) &= \eta_X \left[\pi(x|y) - \delta\left[W(x|y) - V(\tilde{x}|\tilde{y})\right] - r_X c_X\right] \end{aligned}$$

where the second line draws on equations (5) and (6). From equation (4), we infer

$$W(x|y) = \frac{r_X + \eta_X}{\eta_X} V(\tilde{x}|\tilde{y}) + c_X$$

which allows us to substitute $W(x|y)$ out:

$$\begin{aligned}
(r_X + \eta_X) r_X V(\tilde{x}|\tilde{y}) &= \eta_X \left[\pi(x|y) - \delta \left[\frac{r_X}{\eta_X} V(\tilde{x}|\tilde{y}) + c_X \right] - r_X c_X \right] \\
&= \eta_X \left[\pi(x|y) - \delta c_X - r_X c_X \right] - \delta r_X V(\tilde{x}|\tilde{y}) \\
\Leftrightarrow r_X V(\tilde{x}|\tilde{y}) &= \frac{\eta_X}{r_X + \delta + \eta_X} \left[\pi(x|y) - c_X (r_X + \delta) \right] \qquad (8)
\end{aligned}$$

By analogy also

$$r_Y V(\tilde{y}|\tilde{x}) = \frac{\eta_Y}{r_Y + \delta + \eta_Y} \left[\pi(y|x) - c_Y (r_Y + \delta) \right] \tag{9}$$

Hence the total surplus agents derive from match output is a strictly positive expression:

$$f(x, y) - r_X V(\tilde{x}|\tilde{y}) - r_Y V(\tilde{y}|\tilde{x}) = \left[\pi(x|y) - r_X V(\tilde{x}|\tilde{y}) \right] + \left[\pi(y|x) - r_Y V(\tilde{y}|\tilde{x}) \right] \tag{10}$$

However, to ensure that agents are willing to engage in search and attend meetings in the first place, we also need $V(\tilde{x}|\tilde{y}) \geq 0$ and $V(\tilde{y}|\tilde{x}) \geq 0$. Lemma 1 below takes up this issue. Here we only solve for shares $\pi(x|y)$ and $\pi(y|x)$. It is not clear a priori how the surplus is to be split, and thus agents bargain. The solution to this bargaining is given here by the generalised Nash bargaining solution (generalised NBS), i.e. the combination of shares that maximises the weighted product of the individual surpluses:

$$\arg \max_{\pi(x|y), \pi(y|x)} \left[\pi(x|y) - r_X V(\tilde{x}|\tilde{y}) \right]^q \left[\pi(y|x) - r_Y V(\tilde{y}|\tilde{x}) \right]^{1-q}, \tag{11}$$

recalling that $q \in (0, 1)$ represents the bargaining power of any agent $x \in \Theta$. Once $\pi(y|x)$ has been replaced by $f(x, y) - \pi(x|y)$ in line with equation (7), the first-order condition with respect to $\pi(x|y)$ returns

$$\frac{1 - q}{q} = \frac{f(x, y) - \pi(x|y) - r_Y V(\tilde{y}|\tilde{x})}{\pi(x|y) - r_X V(\tilde{x}|\tilde{y})} \tag{12}$$

From this, one obtains the shares in the familiar split-the-surplus formulation:

$$\begin{aligned}
\pi(x|y) &= r_X V(\tilde{x}|\tilde{y}) + q \left[f(x, y) - r_Y V(\tilde{y}|\tilde{x}) - r_X V(\tilde{x}|\tilde{y}) \right] \\
\pi(y|x) = f(x, y) - \pi(x|y) &= r_Y V(\tilde{y}|\tilde{x}) + (1 - q) \left[f(x, y) - r_Y V(\tilde{y}|\tilde{x}) - r_X V(\tilde{x}|\tilde{y}) \right]
\end{aligned}$$

While we will also employ this general formulation in later sections, we take a different route from equation (12) here in order to obtain concise solved-out expressions for the shares. To simplify the exposition, define

$$\psi_X \equiv \frac{\eta_X}{r_X + \delta + \eta_X} c_X (r_X + \delta) \quad \text{and} \quad \psi_Y \equiv \frac{\eta_Y}{r_Y + \delta + \eta_Y} c_Y (r_Y + \delta) \tag{13}$$

Using ψ_X and ψ_Y in equations (8) and (9) and then substituting for $r_X V(\tilde{x}|\tilde{y})$ and $r_Y V(\tilde{y}|\tilde{x})$ in equation (12) gives us

$$\frac{1 - q}{q} = \frac{f(x, y) - \pi(x|y) - \frac{\eta_Y}{r_Y + \delta + \eta_Y} \pi(y|x) + \psi_Y}{\pi(x|y) - \frac{\eta_X}{r_X + \delta + \eta_X} \pi(x|y) + \psi_X} \tag{14}$$

With the substitution $\pi(y|x) = f(x,y) - \pi(x|y)$ once again, this becomes

$$\begin{aligned} \frac{1-q}{q} &= \frac{[f(x,y)-\pi(x|y)]\left[1-\frac{\eta_Y}{r_Y+\delta+\eta_Y}\right]+\psi_Y}{\pi(x|y)\left[1-\frac{\eta_X}{r_X+\delta+\eta_X}\right]+\psi_X} \\ \Leftrightarrow \left[\pi(x|y)\frac{r_X+\delta}{r_X+\delta+\eta_X}+\psi_X\right]\frac{1-q}{q} &= [f(x,y)-\pi(x|y)]\frac{r_Y+\delta}{r_Y+\delta+\eta_Y}+\psi_Y \end{aligned}$$

Dividing through by $\frac{r_Y+\delta}{r_Y+\delta+\eta_Y}$, we rewrite this as

$$\pi(x|y)a = f(x,y) - \pi(x|y) + \frac{r_Y+\delta+\eta_Y}{r_Y+\delta}\psi_Y - \frac{1-q}{q}\frac{r_Y+\delta+\eta_Y}{r_Y+\delta}\psi_X \tag{15}$$

where

$$a \equiv \frac{r_X+\delta}{r_X+\delta+\eta_X}\frac{1-q}{q}\frac{r_Y+\delta+\eta_Y}{r_Y+\delta} \tag{16}$$

With the definitions of ψ_X and ψ_Y above, we note that

$$\frac{r_Y+\delta+\eta_Y}{r_Y+\delta}\psi_Y = \eta_Y c_Y \quad \text{and} \quad \frac{1-q}{q}\frac{r_Y+\delta+\eta_Y}{r_Y+\delta}\psi_X = a\eta_X c_X \tag{17}$$

Hence, finally solving equation (15) for $\pi(x|y)$ returns

$$\pi(x|y) = \frac{1}{1+a}\left[f(x,y) + \eta_Y c_Y - a\eta_X c_X\right] \tag{18}$$

Then $\pi(y|x)$ is derived as follows:

$$\begin{aligned} \pi(y|x) &= f(x,y) - \pi(x|y) \\ &= f(x,y) - \frac{1}{1+a}\left[f(x,y)+\eta_Y c_Y - a\eta_X c_X\right] \\ &= f(x,y)\left(1-\frac{1}{1+a}\right) - \frac{1}{1+a}\left[\eta_Y c_Y - a\eta_X c_X\right] \\ &= f(x,y)\frac{a}{1+a} - \frac{a}{1+a}\left[a^{-1}\eta_Y c_Y - \eta_X c_X\right] \\ &= \frac{1}{1+b}\left[f(x,y)+\eta_X c_X - b\eta_Y c_Y\right] \end{aligned} \tag{19}$$

where

$$b \equiv a^{-1} = \frac{r_Y+\delta}{r_Y+\delta+\eta_Y}\frac{q}{1-q}\frac{r_X+\delta+\eta_X}{r_X+\delta} \tag{20}$$

As one would expect, the share obtained by an agent is increasing in her bargaining power, in the rate at which she can attend meetings, and in the other agent's discount rate (i.e. in the other agent's impatience). It is falling in her own impatience, in the other agent's rate of meetings, in her own explicit costs but increasing in the other agent's explicit costs.

Example 2 When $r_X = r_Y \equiv r$, $\eta_X = \eta_Y \equiv \eta$, and $q = 1-q$, then $a = b = 1$ and the shares given by equations (18) and (19) simplify to

$$\pi(x|y) = \frac{1}{2}\left[f(x,y) + \eta(c_Y - c_X)\right] \quad \text{and} \quad \pi(y|x) = \frac{1}{2}\left[f(x,y) + \eta(c_X - c_Y)\right]$$

When in addition $c_X = c_Y$ or $c_X = c_Y = 0$ as in Shimer and Smith (2000) agents simply split the match output in halves, as is to be expected when agents are in symmetric bargaining positions.

3.3 Participation

The following lemma turns to the problem that explicit costs in the putative equilibrium situation must not be so high that some agents refuse to engage in search.

Lemma 1 (Cost limits). *No agent will strictly prefer not to engage in search in the putative equilibrium situation if the following conditions both hold:*

$$c_X \leq \frac{f(\underline{x}, \underline{y}) + \eta_Y c_Y}{(r_X + \delta)(1 + a) + a\eta_X}, \qquad c_Y \leq \frac{f(\underline{x}, \underline{y}) + \eta_X c_X}{(r_Y + \delta)(1 + b) + b\eta_Y}$$

where $\underline{x} = min_{x \in \Theta} x$ *and* $\underline{y} = min_{y \in \Theta} y$.

Proof. Consider only types $\underline{x}$ and $\underline{y}$ for the moment. The equilibrium match that would normally occur between these types depends on both types participating in the market and engaging in search. To do so, agent $\underline{x}$ requires

$$V(\tilde{\underline{x}}|\tilde{\underline{y}}) \geq 0 \quad \text{where} \quad \underline{x} = x = \underline{y}.$$

Focus on the case when $\underline{x}$ is indifferent, i.e. $V(\tilde{\underline{x}}|\tilde{\underline{y}}) = 0$. To keep the algebra tractable, we begin by finding the output shares that $\underline{x}$ and $\underline{y}$ obtain in this particular case. The first-order condition for the NBS, corresponding to equation (14), becomes:

$$\frac{1-q}{q} = \frac{f(\underline{x}, \underline{y}) - \pi(\underline{x}|\underline{y}) - \frac{\eta_Y}{r_Y + \delta + \eta_Y} \pi(\underline{y}|\underline{x}) + \psi_Y}{\pi(\underline{x}|\underline{y})}$$

Using $\pi(\underline{y}|\underline{x}) = f(\underline{x}, \underline{y}) - \pi(\underline{x}|\underline{y})$ as usual leads to

$$\begin{aligned}
\pi(\underline{x}|\underline{y}) \frac{1-q}{q} &= \left[f(\underline{x}, \underline{y}) - \pi(\underline{x}|\underline{y})\right] \frac{r_Y + \delta}{r_Y + \delta + \eta_Y} + \psi_Y \\
\Leftrightarrow \pi(\underline{x}|\underline{y}) \frac{1-q}{q} \frac{r_Y + \delta + \eta_Y}{r_Y + \delta} &= f(\underline{x}, \underline{y}) - \pi(\underline{x}|\underline{y}) + \frac{r_Y + \delta + \eta_Y}{r_Y + \delta} \psi_Y \\
\Leftrightarrow \pi(\underline{x}|\underline{y}) &= \frac{f(\underline{x}, \underline{y}) + \eta_Y c_Y}{1 + \frac{1-q}{q} \frac{r_Y + \delta + \eta_Y}{r_Y + \delta}} \qquad (21)
\end{aligned}$$

where the last step uses the first part of equation (17). Next, we deduce from equation (8) that $V(\tilde{\underline{x}}|\tilde{\underline{y}}) = 0$ implies

$$c_X (r_X + \delta) = \pi(\underline{x}|\underline{y})$$

and using our result for $\pi(\underline{x}|\underline{y})$, this says

$$c_X = \frac{f(\underline{x}, \underline{y}) + \eta_Y c_Y}{(r_X + \delta)\left(1 + \frac{1-q}{q} \frac{r_Y + \delta + \eta_Y}{r_Y + \delta}\right)}$$

Note that the denominator can be rewritten as

$$(r_X + \delta) + a(r_X + \delta + \eta_X) = (r_X + \delta)(1 + a) + a\eta_X$$

As equation (8) indicates that $V(\tilde{\underline{x}}|\tilde{\underline{y}})$ is strictly decreasing in c_X, we can conclude that

$$c_X \leq \frac{f(\underline{x}, \underline{y}) + \eta_Y c_Y}{(r_X + \delta)(1 + a) + a\eta_X} \qquad (22)$$

is needed for $V(\tilde{\underline{x}}|\tilde{\underline{y}}) \geq 0$. A similar reasoning applies to $\underline{y}$. If $V(\tilde{\underline{y}}|\tilde{\underline{x}}) = 0$, then

$$\pi(\underline{y}|\underline{x}) = \frac{f(\underline{x},\underline{y}) + \eta_X c_X}{1 + \frac{q}{1-q}\frac{r_X+\delta+\eta_X}{r_X+\delta}}$$

and $V(\tilde{\underline{y}}|\tilde{\underline{x}}) \geq 0$ requires

$$c_Y \leq \frac{f(\underline{x},\underline{y}) + \eta_X c_X}{(r_Y+\delta)(1+b) + b\eta_Y} \tag{23}$$

Finally, if c_X and c_Y respectively meet the conditions (22) and (23) when the match output is $f(\underline{x},\underline{y})$, they will always meet these conditions when the match output exceeds $f(\underline{x},\underline{y})$. As the match output in all other equilibrium matches must exceed $f(\underline{x},\underline{y})$, no agent faces prohibitive search costs whenever conditions (22) and (23) are satisfied. □

As an aside, the share $\pi(\underline{x}|\underline{y})$ derived in equation (21) will be weakly smaller than the share in equation (18) if and only if $V(\tilde{\underline{x}}|\tilde{\underline{y}}) \geq 0$ (i.e. equation (22) holds): in the presence of explicit search costs, $V(\tilde{\underline{x}}|\tilde{\underline{y}})$ may be negative, and in such a case $\underline{x}$ will be better off if $V(\tilde{\underline{x}}|\tilde{\underline{y}}) = 0$.

Section 6 will find that the separating equilibrium depends on there being equal numbers of types on both sides of the market. This is the case in our set-up when the conditions in lemma 1 are met, so that all types participate. That these conditions are indeed met is henceforth treated as a part of the putative equilibrium situation. In other words, we analyse behaviour in the putative equilibrium situation below under the premise that all agents participate. Should some types be discouraged by high explicit costs, however, the putative equilibrium situation still applies to the participating types, and all results go through, as long as the numbers of participating types are still equal.

Example 3 When $r_X = r_Y \equiv r$, $\eta_X = \eta_Y \equiv \eta$, $q = 1-q$, and also $c_X = c_Y \equiv c$ the conditions on c_X and c_Y in lemma 1 each simplify to a single condition on c,

$$c(2(r+\delta)+\eta) \leq f(\underline{x},\underline{y}) + \eta c \quad \Leftrightarrow \quad c \leq \frac{f(\underline{x},\underline{y})}{2(r+\delta)}$$

so that the share obtained after the first meeting, discounted at effective discount rate $r+\delta$, needs to outweigh the explicit costs of this one meeting.

4 Equilibrium strategies

Recall that the putative equilibrium situation we described in the previous section consists of three main building blocks: truthful signalling, individual strategies for meeting and stopping, and PPAM. Beginning with the simplest building block, this section will address individual strategies, given truthful signalling and PPAM.

We turn first to agents' meeting strategies. Consider an agent $x \in \Theta$ who sequentially searches the signals sent by agents with a type $y \in \Theta$. Recall that the present value of this initial stage of the search is $U(x)$, and recall that the present value when she seeks to meet an agent signalling $\tilde{y}$ while she herself signals her type as $\tilde{x}$ is denoted $V(\tilde{x}|\tilde{y})$. For any observed signal $\tilde{y} \in \Theta$, the agent will switch from $U(x)$ to $V(\tilde{x}|\tilde{y})$ only if $V(\tilde{x}|\tilde{y}) \geq U(x)$. As she seeks to meet agents in accordance with her meeting strategy $D(x)$, this switch implicitly determines her meeting strategy: a switch from $U(x)$ to $V(\tilde{x}|\tilde{y})$ indicates that $\tilde{y} \in D(x)$. Our task is now to identify the optimal meeting strategy of agent x. With σ_X denoting the rate at which x encounters signals of agents y, we can state the following asset equation:

$$r_X U(x) = \sigma_X \left[\sum_{\tilde{y} \in \Omega(x)} \max[V(\tilde{x}|\tilde{y}), U(x)] l_Y - U(x) \right] \tag{24}$$

The logic behind equation (24) is as follows. The return on the state of search among signals equals the potential gain that comes along with looking at signals, which arrive at rate σ_X. For each $\tilde{y}$, either $\tilde{y} \in D(x)$ or $\tilde{y} \notin D(x)$. Whenever $\tilde{y} \in D(x)$, agent x would like to meet the agent who is sending the signal $\tilde{y}$, implying $\max[V(\tilde{x}|\tilde{y}), U(x)] = V(\tilde{x}|\tilde{y})$. Agent x thereby incurs a gain of $V(\tilde{x}|\tilde{y}) - U(x)$. Whenever $\tilde{y} \notin D(x)$, nothing changes, so that the gain is 0. In principle, x can decide for any $\tilde{y} \in \Theta$ whether she would like a meeting. However, if she seeks a meeting when $\tilde{y} \notin \Omega(x)$ while signalling her type truthfully, she will be turned down straight away and be left with $U(x)$, with no time lost because agents' communication only takes negligible amounts of time (see section 2.2).[10] Formally, $V(\tilde{x}|\tilde{y}) = U(x)$ for all $\tilde{y} \notin \Omega(x)$. Then the gain is 0 and we can leave such signals out of the equation altogether. To express the expected gain under the meeting strategy $D(x)$, the expectation is thus simply taken over all $\tilde{y} \in \Omega(x)$, where we have put l_Y instead of the probability of $\tilde{y}$ because $\tilde{y} = y$ under truthful signalling.

If other agents' signals are encountered at rate σ_X, the time it takes in expectation before another signal arrives is $\frac{1}{\sigma_X}$. Of course, we have already clarified in section 2.2 that observing signals only takes a negligible amount of time. This implies not only that every signal is considered only very briefly, but also that another signal is encountered more or less immediately. Formally, we thus require that $\frac{1}{\sigma_X} \to 0$, or $\sigma_X \to \infty$. Rewriting equation (24) first as

$$\frac{r_X U(x)}{\sigma_X} = \sum_{\tilde{y} \in \Omega(x)} \max[V(\tilde{x}|\tilde{y}), U(x)] l_Y - U(x)$$

we then take the limit:

$$\begin{aligned} \lim_{\sigma_X \to \infty} \frac{r_X U(x)}{\sigma_X} = 0 \quad &\Rightarrow \quad \sum_{\tilde{y} \in \Omega(x)} \max[V(\tilde{x}|\tilde{y}), U(x)] l_Y = U(x) \\ &\Leftrightarrow \quad \sum_{\tilde{y} \in \Omega(x)} \max[V(\tilde{x}|\tilde{y}) - U(x), 0] l_Y = 0 \end{aligned} \tag{25}$$

While these steps may appear to depend crucially on continuous time, an effectively equivalent argument can in fact be made for discrete time.[11] Next, let us focus on the

[10] By such trial and error through costless communication, agents can learn their opportunity set in no time at all. Alternatively, it can be argued that agents understand the equilibrium and infer their opportunity set, and the asset equations in this section therefore presuppose that agents know their opportunity set. To instead make learning through trial and error explicit, the possibility of recall would have to be introduced much like it is in Weitzman (1979).

[11] The key is to note that the Bellman equation for $U(x)$ in discrete time would not include a discount factor on the right-hand side, as arbitrarily many signals may be encountered and considered within one time period.

potentially largest addend on the left-hand side of equation (25), i.e. on

$$\max\left[\max_{\tilde{y}\in\Omega(x)} V(\tilde{x}|\tilde{y}) - U(x), 0\right] l_Y.$$

Suppose we had

$$U(x) < \max_{\tilde{y}\in\Omega(x)} V(\tilde{x}|\tilde{y}).$$

Then the largest addend would be strictly positive. It could not be offset by any strictly negative addend since all addends are at least 0, so that the sum would be strictly positive and violate equation (25). Instead, as Adachi (2003) observed, it will hold if

$$U(x) \geq \max_{\tilde{y}\in\Omega(x)} V(\tilde{x}|\tilde{y})$$

as now $\max[V(\tilde{x}|\tilde{y}) - U(x), 0] = 0$ always. Next, note that $\max_{\tilde{y}\in\Omega(x)} V(\tilde{x}|\tilde{y})$ is the highest present value that agent x can possibly achieve as a direct result of her search among signals, given her opportunity set. Hence, the present value of search among signals cannot exceed this, and so we conclude that

$$U(x) = \max_{\tilde{y}\in\Omega(x)} V(\tilde{x}|\tilde{y}). \tag{26}$$

Because signals are, like types, discretely distributed $\max_{\tilde{y}\in\Omega(x)} V(\tilde{x}|\tilde{y})$ exists and is associated with a unique $\tilde{y}$. Equation (26) then implies that x switches from $U(x)$ to $V(\tilde{x}|\tilde{y})$ only in response to a unique $\tilde{y}$. In other words, the meeting strategy of x must be a singleton, i.e. $|D(x)| = 1$:

$$D(x) = \{\tilde{y} : V(\tilde{x}|\tilde{y}) = \max_{\tilde{y}\in\Omega(x)} V(\tilde{x}|\tilde{y})\} \tag{27}$$

Given that signals are truthful, the optimal meeting strategy is thus to seek meetings exclusively with the agents whose signal is the most promising in the opportunity set. As a more intuitive explanation, an agent will stop searching among signals when the expected marginal benefit of continued search is less than or equal to the expected marginal cost. Section 2.2 specified that the time and costs involved in observing and evaluating signals are negligible, so that the marginal cost of continued search is zero. Hence, the agent will find it optimal to continue searching until the expected marginal benefit is zero as well, i.e. until the most promising signal in the opportunity set is encountered. Note that these conclusions do not depend on the particular approach we have taken. We have set up the choice of meeting strategy as a problem of sequential search among signals because that seems a good description of reality to us. For example, search among advertisements or profiles in an online database proceeds sequentially, even if an automated query returns a selection of non-trivial size. Yet, if one instead assumed that agents observe all current signals at once, the result for the meeting strategy would be the same.

On the background of equation (26), the optimal stopping rule $S(x)$ is quickly determined. If x is already in a meeting, in the putative equilibrium situation she will obtain the payoff $U(x)$ as specified above simply by not accepting the match. For x to accept, her payoff will have to (weakly) exceed this, and so $S(x)$ is found as

$$S(x) = \{y : W(x|y) \geq U(x)\} = \{y : W(x|y) \geq \max_{\tilde{y}\in\Omega(x)} V(\tilde{x}|\tilde{y})\}. \tag{28}$$

The optimal meeting strategy and stopping rule for any agent $y \in \Theta$ are exactly analogous. This section has thus shown that each agent desires a meeting only with agents who send the most promising signal in her opportunity set, given truthful signalling. Moreover, the stopping rules imply that every meeting then results in a match. Both results confirm what we proposed as part of the putative equilibrium situation and will be specified further when we concisely state the entire equilibrium in section 7.1.

5 Truthful signalling

5.1 Agents' preferences over matches

In this section and the next, truthful signalling is placed under scrutiny, taking the individual strategies and PPAM as given. We proceed in two steps: this section explores agents' preferences over matches, while the next section employs the results to examine whether any agent has a unilateral incentive to deviate from truthful signalling in the putative equilibrium situation. We will find that there will be no such incentive to deviate if the degree of supermodularity is sufficiently high.

The reason why we need to worry about false signals in the first place is the existence of search frictions. For an agent who is bargaining with another agent, search frictions make switching to another bargaining partner costly and thereby give market power to the agent she is currently bargaining with. Agents may send false signals in order to manoeuvre themselves into this very position of market power, and then exploit this power to ensure that they are accepted by higher types who would normally not even want to meet them. For example, consider an agent with a rather high type y_H who matches with an exactly corresponding type x_H under truthful signalling. If y_H finds she has been lured into a meeting with a type $x_L < x_H$ by a false signal, she will nevertheless grudgingly accept whenever x_L is not so far below x_H that the costs of another meeting would be justified. Therefore, there can in principle be an incentive to send false signals.

However, we demonstrate below that agents will actually not desire matches with higher types for sufficient degrees of supermodularity γ, and will thus have no incentive to overstate their type. This must appear counterintuitive: after all, $f(x, y)$ is increasing in both arguments irrespectively of the degree of supermodularity, so that higher is better in this sense. Yet we show that this logic does not carry over to bargaining situations: higher types also have higher threat points in bargaining. Therefore, an agent who matches with a high type may end up with a lower share than in a match with some low type.

Our task is to show that each agent prefers her match in the putative equilibrium situation (henceforth the *equilibrium match*) to any alternative match. We consider consecutively three classes of alternative matches: "matches with no-one" (i.e. the agent remains unmatched for the expected duration of a match), matches with higher types, and matches with lower types. In the spirit of the one-deviation principle, we always assume agents to consequently revert to the behaviour prescribed by the putative equilibrium situation. The next section will draw on these results to analyse the incentives to deviate from truthful signalling. For the first class of alternative matches we find:

Lemma 2 (Matches with no-one). *No agent in the putative equilibrium situation prefers being unmatched to her equilibrium match, but each agent strictly prefers her equilibrium match in all but extreme cases (when $q \to 0$ or $q \to 1$).*

Proof. Consider some agent $x \in \Theta$. Recall the present value of her equilibrium match from equation (5) as

$$\begin{aligned} r_X W(x|y) &= \pi(x|y) - \delta[W(x|y) - U(x)] \\ \Leftrightarrow W(x|y) &= \frac{\pi(x|y) + \delta U(x)}{r_X + \delta}, \end{aligned} \tag{29}$$

deliberately leaving $U(x)$ in the equation. Write $W(x|0)$ for the present value of a match with no-one, that is, remaining unmatched for the expected duration $1/\delta$. Also write $\pi(x|0)$ for the flow payoff during this time; of course, $\pi(x|0) = 0$. Then

$$\begin{aligned} r_X W(x|0) &= \pi(x|0) - \delta[W(x|0) - U(x)] \\ \Leftrightarrow W(x|0) &= \frac{\pi(x|0) + \delta U(x)}{r_X + \delta} \end{aligned} \tag{30}$$

Equations (29) and (30) imply

$$W(x|y) \geq W(x|0) \quad \Leftrightarrow \quad \pi(x|y) \geq \pi(x|0) = 0. \tag{31}$$

Hence preference for the equilibrium match requires

$$\pi(x|y) = \frac{1}{1+a}\left[f(x,y) + \eta_Y c_Y - a\eta_X c_X\right] \;\geq\; 0 \tag{32}$$

$$\Leftrightarrow f(x,y) + \eta_Y c_Y \;\geq\; a\eta_X c_X \tag{33}$$

If $a = 0$, equation (33) will hold, for any c_Y. If $a > 0$, this equation requires

$$c_X \leq \frac{f(x,y) + \eta_Y c_Y}{a\eta_X} \tag{34}$$

Now note that since x participates, c_X must weakly satisfy equation (22). Then it will strictly satisfy equation (34) because $(r_X + \delta)(1 + a) > 0$. We conclude that equation (32) holds for any agent x who participates, proving a weak preference for the equilibrium match. The preference is strict whenever $a < \infty$ ($a \geq 0$ by definition). Since all other parameters are bounded away from 0 by definition, $a \to \infty$ if and only if $q \to 0$. Similarly, $b \geq 0$ implies that a agent $y \in \Theta$ has always at least a weak preference for the equilibrium match. It is a strict preference unless $b \to \infty$ due to $q \to 1$. □

Next, we focus on matches with higher types. Without loss of generality, let us take the perspective of some agent with a type $x_L < \max_{x\in\Theta} x$, so that higher types necessarily exist. We want to compare being matched with an exactly corresponding type $y_L = x_L$, as in the equilibrium match, to being matched with a higher type $y_H > x_L$. The comparison $W(x_L|y_L) > W(x_L|y_H)$ reduces to $\pi(x_L|y_L) > \pi(x_L|y_H)$ along the lines of equations (29) through (31), where $\pi(x_L|y_H)$ is the share obtained by x_L in a match with y_H. In order to determine this share, we have to begin with the first-order condition for the NBS again.[12] The equivalent of equation (12) is in this case:

$$\frac{1-q}{q} = \frac{f(x_L, y_H) - \pi(x_L|y_H) - r_Y V(\tilde{y}_H|\tilde{x}_H)}{\pi(x_L|y_H) - r_X V(\tilde{x}_L|\tilde{y}_L)} \tag{35}$$

[12] We cannot simply plug $f(x_L, y_H)$ into equation (18), as its derivation relied on match output being the same in the current match and in following matches, due to the stationarity of an equilibrium situation. When we consider a deviation in the spirit of the one-deviation principle, this no longer holds.

where $V(\tilde{y}_H|\tilde{x}_H)$ indicates that y_H would otherwise match with an exactly corresponding type x_H at the next opportunity and $V(\tilde{x}_L|\tilde{y}_L)$ indicates that x_L would otherwise match with an exactly corresponding type y_L at the next opportunity, as prescribed by the putative equilibrium situation.[13] In order to solve for $\pi(x_L|y_H)$, rewrite equation (35) as

$$(1-q)\left[\pi(x_L|y_H) - r_X V(\tilde{x}_L|\tilde{y}_L)\right] = q\left[f(x_L, y_H) - \pi(x_L|y_H) - r_Y V(\tilde{y}_H|\tilde{x}_H)\right].$$

We then rearrange to obtain the familiar split-the-surplus form

$$\begin{aligned}\pi(x_L|y_H) &= (1-q) r_X V(\tilde{x}_L|\tilde{y}_L) + q\left[f(x_L, y_H) - r_Y V(\tilde{y}_H|\tilde{x}_H)\right] \\ &= r_X V(\tilde{x}_L|\tilde{y}_L) + q\left[f(x_L, y_H) - r_Y V(\tilde{y}_H|\tilde{x}_H) - r_X V(\tilde{x}_L|\tilde{y}_L)\right] \qquad (36)\end{aligned}$$

Let us contrast this with the share obtained in an equilibrium match, also written out in the split-the-surplus form here for convenience:

$$\pi(x_L|y_L) = r_X V(\tilde{x}_L|\tilde{y}_L) + q\left[f(x_L, y_L) - r_Y V(\tilde{y}_L|\tilde{x}_L) - r_X V(\tilde{x}_L|\tilde{y}_L)\right]$$

Agent x_L strictly prefers the match with y_L if $\pi(x_L|y_L) > \pi(x_L|y_H)$, i.e.

$$\begin{aligned}f(x_L, y_L) - r_Y V(\tilde{y}_L|\tilde{x}_L) &> f(x_L, y_H) - r_Y V(\tilde{y}_H|\tilde{x}_H) \\ \Leftrightarrow r_Y V(\tilde{y}_H|\tilde{x}_H) - r_Y V(\tilde{y}_L|\tilde{x}_L) &> f(x_L, y_H) - f(x_L, y_L) \qquad (37)\end{aligned}$$

Equation (37) has a straightforward interpretation. The right-hand side is the increase in total match output that results when x_L matches with y_H instead of y_L. The left-hand side is the difference in the threat points of agents y_H and y_L. With our result for $\pi(y|x)$ from equation (19), we find the threat points of y_H and y_L from equation (9) as

$$\begin{aligned}r_Y V(\tilde{y}_H|\tilde{x}_H) &= \frac{\eta_Y}{r_Y + \delta + \eta_Y}\left[\frac{1}{1+b}\left[f(x_H, y_H) + \eta_X c_X - b\eta_Y c_Y\right] - c_Y(r_Y + \delta)\right] \qquad (38) \\ r_Y V(\tilde{y}_L|\tilde{x}_L) &= \frac{\eta_Y}{r_Y + \delta + \eta_Y}\left[\frac{1}{1+b}\left[f(x_L, y_L) + \eta_X c_X - b\eta_Y c_Y\right] - c_Y(r_Y + \delta)\right] \qquad (39)\end{aligned}$$

When we subtract the latter from the former, all terms involving explicit costs cancel, and thus we rewrite equation (37) as

$$\frac{\eta_Y}{r_Y + \delta + \eta_Y}\frac{1}{1+b}\left[f(x_H, y_H) - f(x_L, y_L)\right] > f(x_L, y_H) - f(x_L, y_L) \qquad (40)$$

It is now clear that the left-hand side of equation (37) is weakly positive (strictly for $b < \infty$, i.e. $q < 1$) because y_H would obtain more match output in her equilibrium match than y_L. Recall from the definition of surplus in equation (10) that a higher match output increases the surplus, while a higher $r_Y V(\tilde{y}|\tilde{x})$ reduces it. Whenever the reduction outweighs the increase, so that equation (37) holds, the surplus in a match between x_L and y_H is unambiguously lower than in a match between x_L and y_L, since $r_X V(\tilde{x}|\tilde{y})$ remains constant. Such an overall fall in surplus renders a match with y_H less preferred, as equation (37) claims, because x_L still obtains the same proportion q of the surplus, as can be seen from the formulation in equation (36). Hence, if the surplus falls while $r_X V(\tilde{x}|\tilde{y})$ is constant, her share will also fall, leaving her worse off than in her equilibrium match. Intuitively, y_H receives additional compensation for her higher opportunity costs, which may outweigh the marginal contribution by y_H vis-à-vis y_L to match output. This explains why higher is not necessarily better here.

[13] As mentioned before, this follows the one-deviation principle: we do not consider any other $V(\tilde{x}|\tilde{y})$ instead of $V(\tilde{x}_L|\tilde{y}_L)$ nor any $V(\tilde{y}|\tilde{x})$ instead of $V(\tilde{y}_H|\tilde{x}_H)$, as these could only arise from a second deviation from the putative equilibrium situation.

Example 4 When $r_X = r_Y \equiv r$, $\eta_X = \eta_Y \equiv \eta$, and $q = 1 - q$ as in Shimer and Smith (2000), equation (40) is

$$\begin{aligned} & \frac{\eta}{r+\delta+\eta}\frac{1}{2}\left[f(x_H,y_H)-f(x_L,y_L)\right] > f(x_L,y_H)-f(x_L,y_L) \\ \Leftrightarrow \quad & \frac{\eta}{r+\delta+\eta} > \frac{2\left[f(x_L,y_H)-f(x_L,y_L)\right]}{f(x_H,y_H)-f(x_L,y_L)}. \end{aligned} \tag{41}$$

It is easily shown that the right-hand side, as the left-hand side, is strictly less than one.[14] Hence, condition (41) can in principle be met and is met the more easily the greater η relative to $r+\delta$. Intuitively, the higher r, the less agents care about shares from alternative matches that could only take place in the future, so that alternative matches (and changes therein) have less impact on current bargaining. An increase in η brings the alternative match closer and thus has the opposite effect (which becomes weaker as agents value time savings less, i.e. as r falls).

Having explored why an agent may prefer her equilibrium match to a match with a higher type, we now turn to the degree of supermodularity needed for this to be indeed the case.

Lemma 3 (Matches with higher types). *In the putative equilibrium situation, any agent $x \in \Theta$ will strictly prefer the equilibrium match to a match with a higher type if the degree of supermodularity γ obeys*

$$\gamma \geq \frac{1 + q\frac{\eta_X}{r_X+\delta}}{(1-q)\frac{\eta_Y}{r_Y+\delta}}$$

Proof. As shown, a match with a higher type will not be preferred to the equilibrium match by any agent $x_L < \max_{x\in\Theta} x$ if equation (40) holds, i.e.

$$\frac{\eta_Y}{r_Y+\delta+\eta_Y}\frac{1}{1+b}f(x_H,y_H) > f(x_L,y_H) - \left[1 - \frac{\eta_Y}{r_Y+\delta+\eta_Y}\frac{1}{1+b}\right]f(x_L,y_L)$$

which we then expand by subtracting the same term in $f(x_L, y_H)$ on both sides:

$$\frac{\eta_Y}{r_Y+\delta+\eta_Y}\frac{1}{1+b}\left[f(x_H,y_H)-f(x_L,y_H)\right] > \tag{42}$$

$$\left[1 - \frac{\eta_Y}{r_Y+\delta+\eta_Y}\frac{1}{1+b}\right]\left[f(x_L,y_H)-f(x_L,y_L)\right] \tag{43}$$

Next, note that we can write

$$f(x_L, y_H) = f(y_L, x_H) \tag{44}$$

[14]The denominator on the right-hand side needs to exceed the numerator:

$$\begin{aligned} f(x_H,y_H)-f(x_L,y_L) &> 2\left[f(x_L,y_H)-f(x_L,y_L)\right] \\ \Leftrightarrow f(x_H,y_H)-f(x_L,y_H) &> f(x_L,y_H)-f(x_L,y_L) \\ \Leftrightarrow f(x_H,y_H)-f(x_L,y_H) &> f(x_H,y_L)-f(x_L,y_L) \end{aligned}$$

where equation (46) allows us to replace $f(x_L, y_H)$ by $f(x_H, y_L)$ in the last step. Of course, the last equation must hold by supermodularity (see assumption 3).

because $x_L = y_L$ and $y_H = x_H$, and in turn

$$f(y_L, x_H) = f(x_H, y_L) \tag{45}$$

by symmetry (see assumption 2). Therefore of course

$$f(x_L, y_H) = f(x_H, y_L) \tag{46}$$

which we use to substitute out $f(x_L, y_H)$ on the right-hand side of equation (42) only. Also dividing through by $\frac{\eta_Y}{r_Y+\delta+\eta_Y}\frac{1}{1+b}$, we are left with

$$f(x_H, y_H) - f(x_L, y_H) > \left[\frac{r_Y + \delta + \eta_Y}{\eta_Y}(1+b) - 1\right][f(x_H, y_L) - f(x_L, y_L)].$$

For this to hold by supermodularity, we need γ to weakly exceed the term in brackets:

$$\begin{aligned}\gamma \geq\ & \frac{r_Y + \delta + \eta_Y}{\eta_Y}(1+b) - 1 = \frac{r_Y + \delta + \eta_Y}{\eta_Y} - 1 + \frac{r_Y + \delta + \eta_Y}{\eta_Y}b = \\ & \frac{r_Y + \delta}{\eta_Y} + \frac{r_Y + \delta}{\eta_Y}\frac{r_X + \delta + \eta_X}{r_X + \delta}\frac{q}{1-q} = \frac{r_Y + \delta}{\eta_Y}\left[1 + \frac{q}{1-q} + \frac{q}{1-q}\frac{\eta_X}{r_X + \delta}\right] = \\ & \frac{r_Y + \delta}{\eta_Y}\frac{1}{1-q}\left[1 + q\frac{\eta_X}{r_X + \delta}\right] = \frac{1 + q\frac{\eta_X}{r_X+\delta}}{(1-q)\frac{\eta_Y}{r_Y+\delta}}\end{aligned} \tag{47}$$

To complete the proof, note that for a type $\bar{x} = \max_{x\in\Theta} x$, a higher type than in the equilibrium match does not exist. $\square$

Lastly, we turn to matches with lower types. Without loss of generality, consider some agent with a type $x_H > \min_{x\in\Theta} x$, so that lower types necessarily exist. For a match between x_H and a type $y_L < x_H$, the analysis is largely analogous to the case of matches with higher types. We will therefore limit ourselves to the important steps. As before, the comparison $W(x_H|y_H) > W(x_H|y_L)$ reduces to $\pi(x_H|y_H) > \pi(x_H|y_L)$. To determine $\pi(x_H|y_L)$, let us take the first-order condition for the NBS:

$$\frac{1-q}{q} = \frac{f(x_H, y_L) - \pi(x_H|y_L) - r_Y V(\tilde{y}_L|\tilde{x}_L)}{\pi(x_H|y_L) - r_X V(\tilde{x}_H|\tilde{y}_H)} \tag{48}$$

Rearranging leads to the split-the-surplus formulation:

$$\pi(x_H|y_L) = r_X V(\tilde{x}_H|\tilde{y}_H) + q\left[f(x_H, y_L) - r_Y V(\tilde{y}_L|\tilde{x}_L) - r_X V(\tilde{x}_H|\tilde{y}_H)\right] \tag{49}$$

As x_H would, in her equilibrium match, obtain a share

$$\pi(x_H|y_H) = r_X V(\tilde{x}_H|\tilde{y}_H) + q\left[f(x_H, y_H) - r_Y V(\tilde{y}_H|\tilde{x}_H) - r_X V(\tilde{x}_H|\tilde{y}_H)\right]$$

she will strictly prefer the equilibrium match if $\pi(x_H|y_H) > \pi(x_H|y_L)$, i.e.

$$f(x_H, y_H) - f(x_H, y_L) > r_Y V(\tilde{y}_H|\tilde{x}_H) - r_Y V(\tilde{y}_L|\tilde{x}_L) \tag{50}$$

$$\text{or}\quad f(x_H, y_H) - f(x_H, y_L) > \frac{\eta_Y}{r_Y + \delta + \eta_Y}\frac{1}{1+b}[f(x_H, y_H) - f(x_L, y_L)] \tag{51}$$

after substitution, where all terms involving explicit costs again cancel out. The left-hand side is the surplus fall due to lower overall match output. The right-hand side is the surplus increase due to the fact that type y_L has lower opportunity costs and thus needs less compensation than y_H. If the former outweighs the latter, the surplus shrinks overall and so does the proportion q of the surplus that x_H obtains. Then a match with a lower type will not be preferred to the equilibrium match. The next question is, what degree of supermodularity is needed for equation (51) to hold.

Lemma 4 (Matches with lower types). *In the putative equilibrium situation, any agent $x \in \Theta$ will strictly prefer the equilibrium match to a match with a lower type if the degree of supermodularity γ obeys*

$$\gamma \geq \frac{(1-q)\frac{\eta_Y}{r_Y+\delta}}{1+q\frac{\eta_X}{r_X+\delta}}$$

Proof. For any agent $x_H > \min_{x\in\Theta} x$, equation (51) applies. From equation (51),

$$\left[1 - \frac{\eta_Y}{r_Y+\delta+\eta_Y}\frac{1}{1+b}\right] f(x_H, y_H) > f(x_H, y_L) - \frac{\eta_Y}{r_Y+\delta+\eta_Y}\frac{1}{1+b} f(x_L, y_L)$$

We now expand by subtracting the same term in $f(x_L, y_H)$ on both sides:

$$\left[1 - \frac{\eta_Y}{r_Y+\delta+\eta_Y}\frac{1}{1+b}\right] [f(x_H, y_H) - f(x_L, y_H)] >$$
$$f(x_H, y_L) - \left[1 - \frac{\eta_Y}{r_Y+\delta+\eta_Y}\frac{1}{1+b}\right] f(x_L, y_H) - \frac{\eta_Y}{r_Y+\delta+\eta_Y}\frac{1}{1+b} f(x_L, y_L)$$

For the first two terms on the right-hand side, the equivalence of $f(x_H, y_L)$ and $f(x_L, y_H)$ implies (see equations (44) through (46)):

$$f(x_H, y_L) - \left[1 - \frac{\eta_Y}{r_Y+\delta+\eta_Y}\frac{1}{1+b}\right] f(x_L, y_H) = \frac{\eta_Y}{r_Y+\delta+\eta_Y}\frac{1}{1+b} f(x_H, y_L)$$

We then divide through by $\frac{\eta_Y}{r_Y+\delta+\eta_Y}\frac{1}{1+b}$ to arrive at

$$\left[\frac{r_Y+\delta+\eta_Y}{\eta_Y}(1+b) - 1\right] [f(x_H, y_H) - f(x_L, y_H)] > f(x_H, y_L) - f(x_L, y_L).$$

By equation (47), the first bracketed term on the left-hand side simplifies to

$$\frac{1+q\frac{\eta_X}{r_X+\delta}}{(1-q)\frac{\eta_Y}{r_Y+\delta}}$$

and hence γ needs to weakly exceed the inverse of this term for supermodularity to imply equation (51). Finally, note that for a type $\underline{x} = \min_{x\in\Theta} x$, a lower type than in the equilibrium match does not exist. □

The condition that matches with higher types are not preferred does not imply the condition that matches with lower types are not preferred, nor vice versa, as is easily shown.[15] Hence, the equilibrium match will be strictly preferred to matches with other

[15]It is legitimate to merge the conditions as stated by equations (37) and (50) into

$$f(x_H, y_H) - f(x_H, y_L) > r_Y V(\tilde{y}_H|\tilde{x}_H) - r_Y V(\tilde{y}_L|\tilde{x}_L) > f(x_L, y_H) - f(x_L, y_L)$$

because the outer inequality holds by our assumption of supermodularity. This merged expression describes a bounded interval on the real line; if one condition implied the other, it would describe an unbounded interval.

types only if both conditions hold. The next section will derive the implications of the entire analysis in this section for agents' signalling behaviour, and will also take into account that corresponding conditions apply from the perspective of some agent $y \in \Theta$.

5.2 Implications for signalling

This section collects the implications of the analysis thus far for agents' signalling behaviour in the putative equilibrium situation. Concretely, we consider one agent's unilateral incentive to deviate from truthful signalling when all other agents signal truthfully. The following proposition shows that, if supermodularity is sufficiently strong, indeed no agent will have such an incentive.

Proposition 1 (Truthful signalling). *Each agent in the putative equilibrium situation weakly prefers (strictly unless $q \to 0$ or $q \to 1$) to signal her type truthfully as long as the degree of supermodularity satisfies*

$$\gamma \geq \max \left[\max \left[\frac{1 + q\frac{\eta_X}{r_X+\delta}}{(1-q)\frac{\eta_Y}{r_Y+\delta}}, \frac{(1-q)\frac{\eta_Y}{r_Y+\delta}}{1 + q\frac{\eta_X}{r_X+\delta}} \right], \max \left[\frac{1 + (1-q)\frac{\eta_Y}{r_Y+\delta}}{q\frac{\eta_X}{r_X+\delta}}, \frac{q\frac{\eta_X}{r_X+\delta}}{1 + (1-q)\frac{\eta_Y}{r_Y+\delta}} \right] \right]$$

Proof. Choose and fix some arbitrary unmatched agent with a type $x \in \Theta$ and call this type x_E. To this exemplary type, a type y_E exactly corresponds. By PPAM, an agent of type x_E is matched with an agent of type y_E unless there is a deviation. As every match is preceded by a meeting, types x_E and y_E will also meet unless there is a deviation. Hence, if x_E does not deviate, but sends a truthful signal $\tilde{x}_E = x_E$, it must be that $\tilde{x}_E \in R(y_E)$. Since $R(y_E) \equiv D(y_E) \cap \Omega(y_E)$, also $\tilde{x}_E \in D(y_E)$. Recalling from equation (27) that $|D(y_E)| = 1$, we have $\tilde{x}'_E \notin D(y_E)$ and also $\tilde{x}'_E \notin R(y_E)$ for any non-truthful signal $\tilde{x}'_E \neq x_E$. If a deviating agent of type x_E thus cannot even meet an agent of type y_E at the next opportunity for a meeting (i.e. after an expected waiting time of $1/\eta_X$), she will have to settle on some alternative at that opportunity. We find it useful to exhaustively categorise the alternatives as follows: (1) meet and match with some type $y \neq y_E$ at the next opportunity,[16] (2) "match with no-one" at the next opportunity, (3) remain inactive at the next opportunity and meet and match with a type $y \in \Theta$ or no-one only at the following opportunity.[17] We address each alternative in order and show that the deviating agent never ends up better off than had she not deviated from truthful signalling. To show this for the first alternative, it suffices to compare the shares $\pi(x_E|y \neq y_E)$ and $\pi(x_E|y_E)$. Now the condition on γ in proposition 1 nests the conditions on γ of both lemma 3 and lemma 4, which therefore apply, together implying a strict preference for the equilibrium match:

$$\pi(x_E|y_E) > \pi(x_E|y \neq y_E) \quad \forall y \neq y_E.$$

For the second alternative, lemma 2 states that the equilibrium match is weakly preferred (strictly unless $q \to 0$ or $q \to 1$) to a match with no-one. For the third alternative, suppose the deviating agent ends up in an equilibrium match at the following opportunity, which

[16]More precisely, she can always only meet with types that exactly correspond to her signal (i.e. $y = \tilde{x}'_E$): any agent of type $y \in \Theta$ only wants to meet exactly corresponding types and takes signals at face value because all agents but the deviant signal truthfully.

[17]Note that letting just one opportunity pass is not the same as a "match with no-one", which has an expected duration of $1/\delta$.

arises at a time t'. Had she not deviated, she would have obtained an equilibrium match at the first opportunity, i.e. at some time $t < t'$. Since $r_X > 0$, this delay will make her strictly worse off.[18] By our examination of the first and second alternatives, any other match at the following opportunity will not make the deviating agent better off than an equilibrium match at the following opportunity, which is itself not a profitable deviation.

Finally, because the agent of type x_E was arbitrarily chosen, all this extends to any agent $x \in \Theta$. The proof for any agent $y \in \Theta$ is exactly analogous, where the reasoning for the first alternative requires that

$$\gamma \geq \max\left[\frac{1+(1-q)\frac{\eta_Y}{r_Y+\delta}}{q\frac{\eta_X}{r_X+\delta}}, \frac{q\frac{\eta_X}{r_X+\delta}}{1+(1-q)\frac{\eta_Y}{r_Y+\delta}}\right]$$

so that the equilibrium match is also strictly preferred to any other match by any agent $y \in \Theta$ (in analogy to lemmas 3 and 4). As the condition in proposition 1 nests this requirement, no agent has a profitable deviation from truthful signalling. $\square$

Looking at the denominators in the condition on γ in proposition 1, it is clear that the condition will very likely fail when $q \to 1$ or $q \to 0$ as two of the fractions will inevitable become very large unless $\frac{\eta_Y}{r_Y+\delta} \to \infty$ or $\frac{\eta_X}{r_X+\delta} \to \infty$, respectively. Proposition 1 thus implies that signals are most unlikely to be truthful whenever the balance of bargaining power is grossly unequal. Note that, when we therefore leave the cases $q \to 1$ or $q \to 0$ out of the picture, the proposition only features *strict* preferences for truthful signals. We do not have to worry, however, about cases where $q = 0$ or $q = 1$ because $q \in (0, 1)$.

Example 5 Suppose again $r_X = r_Y \equiv r$, $\eta_X = \eta_Y \equiv \eta$, and $q = 1 - q$ as in the set-up of Shimer and Smith (2000). The condition in proposition 1 then becomes

$$\begin{aligned} \gamma &\geq \max\left[\max\left[\frac{1+\frac{1}{2}\frac{\eta}{r+\delta}}{\frac{1}{2}\frac{\eta}{r+\delta}}, \frac{\frac{1}{2}\frac{\eta}{r+\delta}}{1+\frac{1}{2}\frac{\eta}{r+\delta}}\right], \max\left[\frac{1+\frac{1}{2}\frac{\eta}{r+\delta}}{\frac{1}{2}\frac{\eta}{r+\delta}}, \frac{\frac{1}{2}\frac{\eta}{r+\delta}}{1+\frac{1}{2}\frac{\eta}{r+\delta}}\right]\right] \\ \Rightarrow \gamma &\geq \frac{1+\frac{1}{2}\frac{\eta}{r+\delta}}{\frac{1}{2}\frac{\eta}{r+\delta}} = \left(\frac{1}{2}\frac{\eta}{r+\delta}\right)^{-1} + 1 = 1 + 2\frac{r+\delta}{\eta} \end{aligned} \tag{52}$$

We can use this to obtain an idea of the numerical range that γ has to lie in. Consider the steady state equation (3) and note that $\Pr(z' \in M(z) | \tilde{z}' \in R(z))$ equals 1 in the putative equilibrium situation, since section 4 showed that every meeting will lead to a match. Then equation (3) implies that $u(z)$ is the same across types (ignoring the subscript Z):

$$\frac{\delta}{\eta} = \frac{u(z)}{l - u(z)} \quad \forall z \in \Theta$$

In most of the developed economies, the right-hand side is small, so that δ should be small relative to η in the real world. In addition, r should be small because agents tend to be patient when searching for long-term matches. In conclusion, $2\frac{r+\delta}{\eta}$ may be fairly small, which illustrates that γ does not necessarily have to be substantially greater than one for proposition 1 to hold in a set-up à la Shimer and Smith (2000). In the light of

[18] In the spirit of the one-deviation principle, we do not consider additional deviations in the future, so that the delay cannot be justified by some future gain vis-à-vis not deviating.

assumption 3, it also suggests that L_X and L_Y would not have to be very discrete.

Proposition 1 concludes an extensive but essentially simple reasoning: if the degree of supermodularity γ is high enough, each agent prefers a match with an exactly corresponding type to all other matches. As the putative equilibrium situation leads to matches with exactly corresponding types as long as signals are truthful, each agent finds it optimal to signal truthfully. The condition in proposition 1 on γ may appear demanding, as if γ had to be on a par with the maximum of four values. In fact, since the paired fractions are inverses of each other, only one of them will exceed unity. Hence, for any choice of parameters, γ has to be on a par with the maximum of only two values, while the remaining two values cannot possibly be a binding constraint on γ.

Nevertheless, the condition in proposition 1 is the most immediate restriction on the entire equilibrium we propose here. But note that the condition ensures more than is necessary for equilibrium: lemmas 3 and 4 derive conditions under which agents prefer the equilibrium match to matches with any higher and any lower types. Yet it would suffice if agents preferred the equilibrium match to matches with those higher or lower types that would, once in a meeting, grudgingly accept them. Preferences for matches with higher or lower types who would not accept such a match are irrelevant, as they will not lead to false signals. Potentially, a weaker condition on γ could therefore achieve the same as the condition in proposition 1, and we hope to present such a condition in a later version of this paper. In any case, we develop a realistic scenario in section 8 in which a weaker condition on γ still ensures truthful signalling.

6 Equilibrium matching

To close the circle, we focus on the overall matching (or "sorting") of agents as the third building block of the putative equilibrium situation. This section shows that PPAM is an equilibrium matching, given truthful signalling and optimal individual strategies, and also that PPAM is even the unique equilibrium matching. Because section 4 concluded that every meeting results in a match when signals are truthful, PPAM with respect to meetings implies PPAM with respect to matches and vice versa. This section can thus focus just on matches without loss of generality.

First we need to clarify what criterion a matching has to meet to be called an equilibrium matching.[19] In Becker's (1973) seminal matching model, a matching is an equilibrium matching whenever it is in the *core*. A standard definition of the core as used by Becker is offered by Telser (1978):

Definition 2 (Core). *Call the set C of agents a coalition, and let $Z(C)$ give the highest possible sum of flow utility the coalition can obtain under the most adverse conditions, with $Z(\emptyset) = 0$. Associated with any matching is a vector $p = (\pi_1, \dots, \pi_n)$ specifying for each matched agent i the flow utility π_i obtained in this matching, where n is the total number of matched agents. The matching is said to be in the core only if*

$$\sum_{i \in C} \pi_i \geq Z(C) \quad \textit{for all legal coalitions } C.$$

[19]Search models usually define equilibrium matchings only indirectly: as matchings that result from each agent pursuing an optimal individual strategy (see, for example, Shimer and Smith (2000) p. 348/49).

In our model, the only legal coalitions are those with $|C| \in \{0, 1, 2\}$, where we also require that one element of N_X and one of N_Y be involved whenever $|C| = 2$. In words, the core is defined as the set of matchings such that no legal coalition can ensure more flow utility for all its members than obtained under the matching. However, this definition of the core is problematic in the context of our model. While an individual agent will only be willing to join a coalition if she thereby personally obtains more utility, it is the sum of utility obtained by the coalition that counts in the definition above. Hence, the standard definition of the core assumes side payments within the coalition. Indeed, side payments are crucial in Becker's (1973) reasoning: an individual agent then always prefers, among all matches available to her, the match generating the highest match output, since her partner in this match will use the extra output to outbid any other potential match partners. Yet in our model output is always divided according to the NBS and an agent's share in the match generating the highest output may fall short of her share in another match. For this reason, Becker's (1973) elegant proof of PPAM as the unique equilibrium matching is no help to us.

Instead, we would have to modify the definition of the core to ensure that each agent's π_i weakly exceeds the utility she obtains in any legal coalition available to her. With the size of legal coalitions limited to two members or less, such a modified definition reduces to two requirements: π_i has to weakly exceed agent i's utility of being single ($|C| = 1$), and no match ($|C| = 2$) is available to i in which she obtains strictly more than π_i. Thanks to our results in section 4, we can identify a match that is available to i with a match where the match partner is better off than in any other available match.[20] Then the modified definition of the core coincides with the standard definition of a *stable matching*:

Definition 3 (Stable matching). *A matching is stable if the associated vector $p = (\pi_1, \ldots, \pi_n)$ of flow utilities satisfies $\pi_i \geq 0$, $\forall i$ and there is no match between any agents x_i and y_j such that $\pi(x_i|y_j) > \pi_i$ and $\pi(y_j|x_i) > \pi_j$.*

Therefore, if the definition of the core is adapted to our model, the set of matchings in the core and the set of stable matchings will coincide, a finding not uncommon in two-sided contexts. A proof that PPAM is the unique stable matching would thus also prove that it is the unique matching in the core of our model, and hence the unique equilibrium matching.

To further prepare precisely such a proof, let us characterise agents' payoffs in equilibrium matchings other than PPAM (in case they exist). Any such matching must include matches between types that do not exactly correspond. Focus on such a match and call the higher type involved x_H and the lower y_L (the analysis for a match between some y_H and x_L is analogous). Given that signals are truthful, x_H must have sought to meet y_L before being matched with her. Moreover, given that each agent's optimal meeting strategy is a singleton, x_H has been willing to meet only y_L. Therefore, we can write the payoff of x_H before she was matched as $V(\tilde{x}_H|\tilde{y}_L)$, where $\tilde{x}_H = x_H$ and $\tilde{y}_L = y_L$. As a consequence of truthful signals, such a match results at the first opportunity:

$$r_X V(\tilde{x}_H|\tilde{y}_L) = \eta_X \left[W(x_H|y_L) - c_X - V(\tilde{x}_H|\tilde{y}_L)\right] \tag{53}$$

[20] The meeting strategies derived in section 4 state that each agent seeks to meet, among the agents she can meet, only those sending the unique most promising signal. Due to truthful signalling, this implies that these agents prefer the resulting matches to any other matches available to them. Hence, when a match is available to i in the sense that i's opportunity set permits the meeting, this match must make the other agent strictly better off than any other available match; otherwise the other agent would not have sought the meeting in the first place.

while in turn

$$r_X W(x_H|y_L) = \pi(x_H|y_L) - \delta\left[W(x_H|y_L) - U(x_H)\right] \tag{54}$$

Now if the matching we are considering here is an equilibrium matching, the stationarity of equilibrium implies that x_H always seeks to meet a type y_L when unmatched, so that

$$U(x_H) = V(\tilde{x}_H|\tilde{y}_L) \tag{55}$$

The same is true whenever bargaining fails, and thus the threat point of agent x_H is

$$r_X V(\tilde{x}_H|\tilde{y}_L) = \frac{\eta_X}{r_X + \delta + \eta_X}\left[\pi(x_H|y_L) - c_X(r_X + \delta)\right] \tag{56}$$

as obtained from equations (53) through (55) by exactly the same algebraic manipulations that determined equation (8). Of course, the threat point of y_L is analogously found as

$$r_Y V(\tilde{y}_L|\tilde{x}_H) = \frac{\eta_Y}{r_Y + \delta + \eta_Y}\left[\pi(y_L|x_H) - c_Y(r_Y + \delta)\right] \tag{57}$$

The NBS for the match between x_H and y_L is

$$\arg\max_{\pi(x_H|y_L),\pi(y_L|x_H)} \left[\pi(x_H|y_L) - r_X V(\tilde{x}_H|\tilde{y}_L)\right]^q \left[\pi(y_L|x_H) - r_Y V(\tilde{y}_L|\tilde{x}_H)\right]^{1-q},$$

and with the threat points found above, taking the first-order condition and applying the same steps as in equations (12) through (19) returns

$$\pi(x_H|y_L) = \frac{1}{1+a}\left[f(x_H, y_L) + \eta_Y c_Y - a\eta_X c_X\right] \tag{58}$$

$$\text{and} \quad \pi(y_L|x_H) = \frac{1}{1+b}\left[f(x_H, y_L) + \eta_X c_X - b\eta_Y c_Y\right]. \tag{59}$$

In short, given that the matching is supposed to be an equilibrium matching, given truthful signalling, and given optimal individual strategies, agents' shares in this matching are exactly analogous to the shares in the putative equilibrium situation. Only match outputs differ. We are now ready to prove the following proposition:

Proposition 2 (Perfect assortative matching). *Given truthful signalling and individual strategies as in the putative equilibrium situation, perfect positive assortative matching is a stable matching and the unique equilibrium matching whenever the degree of supermodularity satisfies the condition in proposition 1.*

Proof. Consider the two top types $\bar{x} = \max_{x\in\Theta} x$ and $\bar{y} = \max_{y\in\Theta} y$, which we call x_H and y_H here for expositional reasons. Suppose there was a stable matching, given truthful signalling and individual strategies as in the putative equilibrium situation, in which x_H and y_H are not matched with each other. As higher types do not exist, both must then be matched with lower types, which we call y_L and x_B with exactly corresponding types x_L and y_B. No particular order is assumed, i.e. $y_L \lesseqgtr y_B$. Focus on the match between x_H and y_L. Recall the argument leading up to lemma 4 in section 5: x_H does not desire such a match if the loss of match output outweighs the fall in the threat point of the other agent, so that the surplus falls. For PPAM, equation (50) gives this condition as

$$f(x_H, y_H) - f(x_H, y_L) > r_Y V(\tilde{y}_H|\tilde{x}_H) - r_Y V(\tilde{y}_L|\tilde{x}_L) \tag{60}$$

$$\text{implying} \quad \gamma \geq \frac{(1-q)\frac{\eta_Y}{r_Y+\delta}}{1+q\frac{\eta_X}{r_X+\delta}}$$

by the proof of lemma 4. We seek a corresponding condition for the matching where x_H and y_H match with lower types instead of PPAM. Suppose x_H, while in a match with y_L, has the opportunity to deviate and match with a type y_H instead. Along the lines of equations (29) through (31), the comparison $W(x_H|y_L) > W(x_H|y_H)$ reduces to the comparison $\pi(x_H|y_H) > \pi(x_H|y_L)$ (because $U(x_H) = V(\tilde{x}_H|\tilde{y}_L)$ in both cases). Writing both shares in the split-the-surplus formulation:

$$\begin{aligned}
\pi(x_H|y_L) &= r_X V(\tilde{x}_H|\tilde{y}_L) + q\left[f(x_H, y_L) - r_Y V(\tilde{y}_L|\tilde{x}_H) - r_X V(\tilde{x}_H|\tilde{y}_L)\right] \\
\pi(x_H|y_H) &= r_X V(\tilde{x}_H|\tilde{y}_L) + q\left[f(x_H, y_H) - r_Y V(\tilde{y}_H|\tilde{x}_B) - r_X V(\tilde{x}_H|\tilde{y}_L)\right]
\end{aligned}$$

In the spirit of the one-deviation principle, the threat points of both agents are given here by continued search for the kind of match they deviate from. Then x_H will not desire the match with y_L but strictly prefer the deviation if $\pi(x_H|y_H) > \pi(x_H|y_L)$, i.e.

$$\begin{aligned}
f(x_H, y_H) - r_Y V(\tilde{y}_H|\tilde{x}_B) &> f(x_H, y_L) - r_Y V(\tilde{y}_L|\tilde{x}_H) \\
\Leftrightarrow f(x_H, y_H) - f(x_H, y_L) &> r_Y V(\tilde{y}_H|\tilde{x}_B) - r_Y V(\tilde{y}_L|\tilde{x}_H)
\end{aligned} \tag{61}$$

The left-hand sides of equations (60) and (61) are the same. As to the right-hand sides, $r_Y V(\tilde{y}_L|\tilde{x}_H)$ is given by combining equations (57) and (59) to

$$r_Y V(\tilde{y}_L|\tilde{x}_H) = \frac{\eta_Y}{r_Y + \delta + \eta_Y}\left[\frac{1}{1+b}\left[f(x_H, y_L) + \eta_X c_X - b\eta_Y c_Y\right] - c_Y(r_Y + \delta)\right] \tag{62}$$

To see how this compares to $r_Y V(\tilde{y}_L|\tilde{x}_L)$, recall equation (39):

$$r_Y V(\tilde{y}_L|\tilde{x}_L) = \frac{\eta_Y}{r_Y + \delta + \eta_Y}\left[\frac{1}{1+b}\left[f(x_L, y_L) + \eta_X c_X - b\eta_Y c_Y\right] - c_Y(r_Y + \delta)\right]$$

Hence we find $r_Y V(\tilde{y}_L|\tilde{x}_H) > r_Y V(\tilde{y}_L|\tilde{x}_L)$ since $f(x_H, y_L) > f(x_L, y_L)$. The same reasoning allows us to conclude $r_Y V(\tilde{y}_H|\tilde{x}_B) < r_Y V(\tilde{y}_H|\tilde{x}_H)$ since $f(x_B, y_H) < f(x_H, y_H)$. Therefore, the right-hand side of equation (61) is unambiguously smaller than that of equation (60). The latter thus implies the former, and while

$$\gamma \geq \frac{(1-q)\frac{\eta_Y}{r_Y+\delta}}{1+q\frac{\eta_X}{r_X+\delta}}$$

is not necessary for equation (61) to hold, it must be sufficient, so that x_H will indeed strictly prefer deviating to a match with y_H if γ meets this condition. By the definition of a stable matching, such a preference will only render a matching instable if it is reciprocated, so that y_H also strictly prefers to be matched with x_H instead of x_B. The reasoning from the perspective of y_H is analogous and arrives at the sufficient condition

$$\gamma \geq \frac{q\frac{\eta_X}{r_X+\delta}}{1+(1-q)\frac{\eta_Y}{r_Y+\delta}} \tag{63}$$

As both conditions are included in the condition in proposition 1, we conclude that the top types x_H and y_H would both strictly prefer to be matched with each other, so that no matching where they are not matched with each other can be stable. Now consider any matching where the two top types are matched with each other but the second-highest types are not. The second-highest types cannot be matched with one of the top types because the top types exclusively match with each other, and must therefore be matched with some lower types. Hence, the entire reasoning we have applied to the two top types now carries over to the second-highest types, yielding the same conditions. Since the set-up of our model results in an equal number of types on both sides of the market, the logic applies successively down to the two bottom types $\underline{x}$ and $\underline{y}$.

In conclusion, whenever there is a match between types that do not exactly correspond, the matching cannot be stable, provided the condition on γ in proposition 1 holds. Then PPAM remains as the only candidate for a stable matching. Lemmas 3, 4 and analogous lemmas for an agent $y \in \Theta$ imply that, under the condition in proposition 1, not even a single agent wishes to be matched with another type in a situation of PPAM. Lemma 2 shows that $\pi_i \geq 0 \; \forall i$, so that no agent prefers to be single either. Hence PPAM is the unique stable matching and therefore the unique equilibrium matching. □

A couple of comments on proposition 2 and its proof are in order. First, PPAM might still be a stable matching even if the condition on γ is not met. However, this is irrelevant for the separating equilibrium because proposition 1 requires this very condition on γ in any case. Second, much of the proof may appear as just a repetition of the proof of lemma 4. Yet the latter examined the unilateral incentive to deviate from truthful signalling, given PPAM, while the former examines the incentives of two agents to deviate from some matching, given truthful signals. Third, a stable matching is a most unusual result in a model with search frictions. In standard search models, agents cannot search selectively and might thus be matched with any type from a certain range of types. Of course, many of these types are only accepted because search frictions make continued search undesirable. A stable matching as in definition 2 cannot be expected to arise under such circumstances and is very unlikely to arise by chance whenever the number of different types is not trivially small. By requiring that an equilibrium matching in our model be a stable matching, we have applied a very demanding criterion that is normally only applied in matching models (i.e. models without search frictions), and that normally only matching models meet.

Lastly, but most importantly, the result of PPAM in a model with search frictions is, to the best of our knowledge, new to the literature. Shimer and Smith (2000) establish assortative matching, albeit not perfect assortative matching, in a model with search frictions in the form of discounting under the condition that the match production function, the logarithm of its first derivative, and the logarithm of its cross-partial derivative are all supermodular. Atakan (2006) replaces discounting in that model by explicit additive search costs. He shows that strict supermodularity as such ensures positive assortative matching, albeit not perfect positive assortative matching as in Becker's (1973) frictionless model. His results crucially depend on search costs being identical for all agents. Our set-up essentially adds signals to the model in Shimer and Smith (2000), maintains discounting, but also allows for differential explicit costs. This set-up achieves PPAM just with sufficiently strong supermodularity of the production function.

With proposition 2 established, each of the three building blocks of the putative equilibrium situation has been examined, taking the two others as given. Our finding is that the putative equilibrium situation is indeed an equilibrium whenever the degree of supermodularity is sufficiently high. The next section recollects this equilibrium.

7 Equilibrium properties

7.1 Complete characterisation of equilibrium behaviour

The previous three sections have each shown one building block of the putative equilibrium situation to hold, given the other two building blocks. This section collects the results and concisely states the separating equilibrium whose existence we have thereby proven, and leads up to a discussion of some equilibrium properties. Reflecting our definition of search equilibrium in definition 1, the PBE entails the following elements:

Steady state. The pointwise steady state was given by equations (1) and (2) but left $Pr(y \in M(x)|\tilde{y} \in R(x))$ and $Pr(x \in M(y)|\tilde{x} \in R(y))$ undetermined. As noted before, both probabilities turn out to equal 1 because every meeting leads to a match (see section 4). Solved for the probabilities of unmatched types, the conditions for the steady state then require

$$u_X(x) = \frac{\delta l_X}{\eta_X + \delta} \quad \forall x \in \Theta \quad \text{and} \quad u_Y(y) = \frac{\delta l_Y}{\eta_Y + \delta} \quad \forall y \in \Theta$$

Signalling. If the degree of supermodularity γ satisfies the condition in proposition 1, each agent $x \in \Theta$ will find it optimal to always signal her type truthfully as $\tilde{x} = x$, and likewise for each agent $y \in \Theta$.

Beliefs. The putative equilibrium situation postulated that each agent believes all other agents to signal truthfully. Formally, $\mu_X(y|\tilde{y}) = 1$ for all $\tilde{y} = y$, and $\mu_Y(x|\tilde{x}) = 1$ for all $\tilde{x} = x$. As all agents do indeed signal truthfully, these beliefs correctly describe equilibrium play, as is required in a PBE. There are no off-equilibrium signals that would necessitate beliefs for such signals.

Matching. If the degree of supermodularity γ satisfies the condition in proposition 1, optimal behaviour will lead agents to meet and match in accordance with PPAM.

Shares and payoffs. In a match between x and y with $x = y$, the share for x is

$$\pi(x|y) = \frac{1}{1+a}\left[f(x, y) + \eta_Y c_Y - a\eta_X c_X\right]$$

where a is given by equation (16). For the payoffs to x in the various states, we found

$$U(x) = V(\tilde{x}|\tilde{y}) = \frac{\eta_X}{r_X(r_X + \delta + \eta_X)}\left[\frac{1}{1+a}\left[f(x, y) + \eta_Y c_Y - a\eta_X c_X\right] - c_X(r_X + \delta)\right]$$

and, using equation (4),

$$W(x|y) = \frac{r_X + \eta_X}{\eta_X} V(\tilde{x}|\tilde{y}) + c_X$$

Analogous results, with b (see equation (20)) replacing a, were obtained for $\pi(y|x)$, $U(y)$, $V(\tilde{y}|\tilde{x})$, and $W(y|x)$.

Individual strategies. As agents correctly believe that they can trust observed signals, each agent only meets with agents who send the most promising signal in her opportunity set. From PPAM we know that the most promising $\tilde{y}$ in the $\Omega(x)$ of any agent $x \in \Theta$ who signals truthfully is $\tilde{y} = y = x$, i.e. the truthful signal of the exactly corresponding type:

$$D(x) = \{\tilde{y} : V(\tilde{x}|\tilde{y}) = \max_{\tilde{y} \in \Omega(x)} V(\tilde{x}|\tilde{y})\} = \{\tilde{y} : \tilde{y} = x\}$$

The strategy $D(y)$ is the mirror image. Hence $\Omega(y) = \{\tilde{x} : \tilde{y} = \tilde{x}\}$ and $\Omega(x) = \{\tilde{y} : \tilde{x} = \tilde{y}\}$. Meetings thus occur only between types x and y where $\tilde{x} = \tilde{y}$ and also $x = y$. The stopping rule is

$$S(x) = \left\{ y : W(x|y) \geq \max_{\tilde{y} \in \Omega(x)} V(\tilde{x}|\tilde{y}) \right\} = \{y : W(x|y) \geq V(\tilde{x}|\tilde{y})\}$$

with the $V(\tilde{x}|\tilde{y})$ just specified, and likewise for $S(y)$. Since $W(x|y) > V(\tilde{x}|\tilde{y})$ for the $W(x|y)$ in equilibrium, all meetings result in matches.

7.2 Uniqueness

While we have shown that the putative equilibrium situation exists as a separating equilibrium (for γ high enough), there might also be other separating equilibria. However, the following result arises as a corollary of propositions 1 and 2:

Corollary 1 (Uniqueness). *The separating equilibrium described by the putative equilibrium situation is the unique separating equilibrium whenever it exists.*

No formal proof is needed, since all elements of the separating equilibrium as stated in the last section turn out to be unique. Least obvious is the uniqueness of the equilibrium matching, which is proven in proposition 2. That apart, the steady state conditions uniquely determine a probability mass for the unmatched agents of each type. Agents have only one way of signalling their types truthfully. Then no other specification of beliefs will be compatible with the fact that all agents signal truthfully. Finally, section 4 shows that each agent's meeting strategy and stopping rule are uniquely determined by the opportunity set. As every meeting leads to a match, a unique equilibrium matching then implies a unique sorting already in the meetings and thereby a unique equilibrium opportunity set for each agent. With all elements of the separating equilibrium being unique, so must be the equilibrium as a whole.

7.3 Efficiency

The unique separating equilibrium we have identified is efficient in a number of important respects. First and foremost, search costs are minimised, both for each agent individually and overall: in equilibrium, truthful signals allow each agent to ensure that no meeting is wasted, but that every meeting she attends results in a match. Hence, whenever an agent x searches, she attends a meeting and also matches at the first opportunity, that is, after an expected search time of $1/\eta_X$. This is the absolute minimum because a meeting necessarily precedes a match. Without truthful signals, x would still attend meetings at rate η_X but would only match whenever y happens to be in the matching set $M(x)$. In a standard search model, each match would typically be preceded by a number of pointless meetings, and only by chance will the first meeting of an agent result in a match. Therefore, search costs in standard search models are at least as high from the individual perspective as in our model with truthful signals, and much higher in expectation as well as on aggregate.

Second, not only is each agent matched at the first opportunity in equilibrium, but each agent is also matched with the type she most prefers among the types in the matching set (as she only meets agents of this type). This again contrasts starkly with standard search models, where the type that an agent x expects in a match is the expectation over $M(x)$, not the most preferred element of it. In fact, matches in our separating equilibrium are even with the type that is most preferred among all types on the market.

Yet this arises here due to sufficiently strong supermodularity: for lower γ, higher or lower types outside $M(x)$ may be the globally most preferred type, instead of the exactly corresponding type $y = x$. The coincidence of the most preferred type in $M(x)$ and the globally most preferred type can therefore not be expected in a standard search model.

Finally, note that all agents match in equilibrium so that there is no unrealised surplus left in the form of unmatched agents. On the contrary, the combination of supermodularity and PPAM maximises aggregate match output; a proof of this can be found in the appendix of Becker (1973). Standard search models, be it with or without supermodularity of the match production function, do generally not maximise aggregate match output, as they lead to a certain degree of mismatch instead of PPAM.

8 Extension: an alternative scenario

We briefly present a scenario in this section that is arguably more realistic than the base model and leads to a weaker condition on the degree of supermodularity. Suppose types are not verifiable. Instead, an agent can deceive potential match partners even in a meeting and can enter into a match undetected. If, however, she is found out at any point during a match, the match will terminate immediately. The reasons for the immediate end of the match in such a case might be of a legal nature, or the match partner might be concerned about her reputation, or the break of trust might make any further co-operation practically impossible. All these reasons are outside our model. A good example of what we have in mind is false doctors: on the basis of forged qualifications, such doctors have often practised illegally for years before they are detected (which suggests that many more cases are never detected). False doctors are rarely detected because they manage in practice to somehow make up for their lack of formal qualifications. Yet whenever they are detected, they are immediately and permanently removed from their positions despite sometimes years of good work, typically for the aforementioned reasons.

We model this as follows. Any agent can signal to be of some type, not necessarily her true type, and can then behave and match just as the emulated type could, provided the adjustment due to the discrepancy between her true type and the emulated type falls entirely on herself. When her true type differs from her signalled type, the actual match output differs from the match output one would expect, given the signals. Adjustment then means that the agent whose signal is false gives up as much of her own share as is necessary to bridge the gap when actual output is lower than the signals would suggest. When actual output is higher she quietly pockets excess output. As the latter would typically be considered fraudulent, a match would thus also terminate if an agent is found out to have a higher type than signalled. In any case, everything seems perfectly normal to her match partner as long as the necessary adjustment is secretly ensured. The deceit will be detected, however, as soon as there is a lack of adjustment. In that moment, the match terminates and both agents join the pool of searchers again.

In this set-up, no agent would admit to a false signal in the meeting. Even if the potential match partner would rather accept the agent despite the false signal than continue searching, this will be out of the question. For the example of false doctors, a potential employer would in legal terms become an accomplice of the false doctor if she nevertheless accepted her for a match. In addition, due to reputational effects, the existence of the entire clinic might be at risk if the employer did accept her. And indeed, false doctors will not be knowingly accepted in practice, even if that appears more opportune than continued search.

We start again from the putative equilibrium situation outlined in section 3 but without explicit costs. An agent who sends a false signal will only agree to the match if the present value of being in this match exceeds the present value of search, which implies that she would then always prefer remaining in the match to recommencing search. Hence an optimising agent will either not deviate from the putative equilibrium situation or will both send a false signal and keep her true type secret in the meeting as well as for the entire match duration. We assume that there is no other way to detect false signals, so that such a match lasts just as long as any other match, a time $1/\delta$ in expectation, unless one agent voluntarily reveals that her signal was false.

To re-examine agents' incentives to deviate from truthful signalling in this scenario, we begin with their preferences over matches. Turning first to matches with higher types, consider a type $x_L < \max_{x\in\Theta} x$. Along the lines of equations (29) through (31), the choice between the equilibrium match with a type y_L and a match with a higher type $y_H > y_L$ reduces to the choice between flow utilities obtained during the match. Her gross share in a match with y_H will be the same as the share a type x_H would obtain, as she will successfully emulate a type x_H. Since x_H and y_H are matched with each other in the putative equilibrium situation, the gross share is given by equation (18) for a match output $f(x_H, y_H)$. Her net share, however, is the flow utility left for her after the adjustment due to lower match output, i.e. after subtracting $f(x_H, y_H) - f(x_L, y_H)$. Hence, x_L will strictly prefer her equilibrium match if

$$\begin{aligned} \pi(x_L|y_L) &> \pi(x_H|y_H) - [f(x_H, y_H) - f(x_L, y_H)] \\ \Leftrightarrow \frac{1}{1+a} f(x_L, y_L) &> \frac{1}{1+a} f(x_H, y_H) - [f(x_H, y_H) - f(x_L, y_H)] \\ \Leftrightarrow f(x_L, y_L) &> (1+a) f(x_L, y_H) - a f(x_H, y_H) \end{aligned} \tag{64}$$

Rearranging gives

$$a\,[f(x_H, y_H) - f(x_L, y_H)] > f(x_H, y_L) - f(x_L, y_L) \tag{65}$$

where we have replaced $f(x_L, y_H)$ on the right-hand side by $f(x_H, y_L)$ due to the symmetry of the match production function. This equation directly implies that x_L will strictly prefer the equilibrium match if $\gamma \geq a^{-1} = b$. From the perspective of a type $y_L < \max_{y\in\Theta} y$, the same approach leads to the condition $\gamma \geq b^{-1} = a$, as is easily verified.

Next, we turn to matches with lower types. A type $x_H > \min_{x\in\Theta} x$ will strictly prefer her equilibrium match if

$$\begin{aligned} \pi(x_H|y_H) &> \pi(x_L|y_L) + [f(x_H, y_L) - f(x_L, y_L)] \\ \Leftrightarrow \frac{1}{1+a} f(x_H, y_H) &> \frac{1}{1+a} f(x_L, y_L) + [f(x_H, y_L) - f(x_L, y_L)] \\ \Leftrightarrow f(x_H, y_H) &> (1+a) f(x_H, y_L) - a f(x_L, y_L) \\ \Leftrightarrow f(x_H, y_H) - f(x_L, y_H) &> a\,[f(x_H, y_L) - f(x_L, y_L)] \end{aligned}$$

where we have replaced $f(x_H, y_L)$ on the left-hand side by $f(x_L, y_H)$. Hence, $\gamma \geq a$ is required, and a similar argument from the perspective of an agent $y_H > \min_{y\in\Theta} y$ leads to $\gamma \geq b$. Altogether, we thus find that any agent $x \in \Theta$ or $y \in \Theta$ who can in principle deviate strictly prefers her respective equilibrium match as long as $\gamma \geq \max[a, b]$. Based on the arguments above and lemma 2 as before (but no longer on lemmas 3 and 4), a straightforward adaptation of proposition 1 and its proof implies that $\gamma \geq \max[a, b]$ ensures truthful signalling in this alternative scenario.

In order to compare the condition $\gamma \geq \max[a, b]$ to the condition on γ stated in proposition 1, consider first a type $x_L < \max_{x\in\Theta} x$ again who might match with higher types. For such types to strictly prefer the equilibrium match, $\gamma \geq b$ is required in this scenario. According to lemma 3, the base model requires

$$\gamma \geq \frac{1 + q\frac{\eta_X}{r_X+\delta}}{(1-q)\frac{\eta_Y}{r_Y+\delta}} = \frac{r_Y + \delta + \eta_Y}{\eta_Y}(1+b) - 1$$

where the equality derives from equation (47). Now since

$$1 + b < \frac{r_Y + \delta + \eta_Y}{\eta_Y}(1+b) \tag{66}$$

the condition in the base model is more restrictive in this example. However, for a type $x_H > \min_{x\in\Theta} x$ who might match with lower types, the inverses apply. Then equation (66) implies that the condition in the alternative scenario is more restrictive this time:

$$a = b^{-1} > \left(\frac{r_Y + \delta + \eta_Y}{\eta_Y}(1+b) - 1\right)^{-1} = \frac{(1-q)\frac{\eta_Y}{r_Y+\delta}}{1 + q\frac{\eta_X}{r_X+\delta}} \tag{67}$$

Importantly, this last observation helps us adapt proposition 2 to the alternative scenario: $\gamma \geq a$ implies that γ is also weakly greater than the last term in equation (67). The latter condition was needed to ensure that equation (60) holds. This equation was shown in the proof of proposition 2 to imply equation (61). The relation must still hold with $\gamma \geq a$, so that, in any matching where the the top types $\bar{x}$ and $\bar{y}$ are not matched with each other, $\bar{x}$ prefers to deviate from a match with some $y_L < \bar{y}$ to the match with $\bar{y}$. We then need to show that $\bar{y}$ would also deviate from her current match with some $x_B < \bar{x}$ to the match with $\bar{x}$. It was found that this is indeed the case whenever equation (63) holds. Since

$$b = a^{-1} > \left(\frac{r_X + \delta + \eta_X}{\eta_X}(1+a) - 1\right)^{-1} = \frac{q\frac{\eta_X}{r_X+\delta}}{1 + (1-q)\frac{\eta_Y}{r_Y+\delta}}$$

the alternative scenario's requirement $\gamma \geq b$ is again more restrictive than the base model and implies equation (63). We can conclude that the reasoning of the proof of proposition 2 is still valid in the alternative scenario: PPAM remains as the only candidate for a stable matching. By lemma 2 and our arguments above, PPAM will be a stable matching if $\gamma \geq \max[a, b]$.

In the alternative scenario, the condition $\gamma \geq \max[a, b]$ thus ensures both truthful signalling and PPAM. While the condition is more restrictive with respect to matches with lower types than in the base model, it appears weaker overall than that in the base model in two ways. Firstly, since a and b are inverses of each other, $\gamma \geq \max[a, b]$ requires γ to be on a par with only one value greater than one, not with two values greater one as in the base model. Secondly, the condition in the alternative scenario is unambiguously weaker in the important special case when $a = b = 1$ as in Shimer and Smith (2000): the alternative scenario then only requires $\gamma \geq 1$, i.e. supermodularity as such, while equation (52) tells us that the base model would require $\gamma \geq 1 + 2\frac{r+\delta}{\eta}$.

9 Non-separating equilibria

9.1 Babbling equilibrium

As is typically the case in cheap-talk models, there may be a *babbling equilibrium* of our basic model in which all agents simply ignore any signal so that signals are irrelevant for the equilibrium. Our model has added signals to a framework close to that in Shimer and Smith (2000); it therefore does not come as a surprise that, when signals are rendered irrelevant in a babbling equilibrium, this equilibrium is close to that in Shimer and Smith (2000). However, Shimer and Smith do not provide solved-out expressions, and the differences between our model and theirs may nevertheless make it hard to see correspondences. Therefore, we fully explore here what a babbling equilibrium would consist of, while we do not investigate existence of such an equilibrium.

By the nature of a babbling equilibrium, neither a signal sent nor a signal received makes any difference to an agent's situation:

$$V(\tilde{x}|\tilde{y}) = U(x) \quad \forall \tilde{x}, \tilde{y}, x \quad \text{and} \quad V(\tilde{y}|\tilde{x}) = U(y) \quad \forall \tilde{y}, \tilde{x}, y$$

Agents still manage to attend meetings at the same rates as before,[21] but since signals cannot guide any agent's search, the types met are now random draws from the market. We follow Shimer and Smith (2000) in assuming that more frequent types on the market will accordingly be met more frequently: the probability of drawing a type y is $u_Y(y)$, and the probability of meeting a type y belonging to the matching set $M(x)$ is $\sum_{y \in M(x)} u_Y(y)$. This leads to the following conditions for a pointwise steady state:

$$\delta[l_X - u_X(x)]|N_X| = \eta_X u_X(x)|N_X| \sum_{y \in M(x)} u_Y(y) \quad \forall x \in \Theta \tag{68}$$

$$\delta[l_Y - u_Y(y)]|N_Y| = \eta_Y u_Y(y)|N_Y| \sum_{x \in M(y)} u_X(x) \quad \forall y \in \Theta \tag{69}$$

When types are ex ante indistinguishable, there are only two rational meeting strategies: either seize on every opportunity for a meeting, or do not engage in search at all. The latter carries a present value of 0, while the former will be weakly positive if explicit costs are not prohibitively high, which we henceforth assume. This immediately implies that we can write the meeting strategy of agent x as

$$D(x) = \{\tilde{y} : \tilde{y} \in \Theta\} \quad \text{i.e.} \quad \tilde{y} \in D(x), \forall \tilde{y} \tag{70}$$

Next, the asset equation for $U(x)$ is:

$$r_X U(x) = \eta_X \sum_{y \in M(x)} [W(x|y) - c_X - U(x)]\, u_Y(y) \tag{71}$$

The summation gives the expected gain in payoffs over all types $y \in M(x)$, as a match will result only in these cases. For any $y \notin M(x)$, the gain is always zero. Meanwhile, the

[21] Effectively, every agent x appears to be acceptable to every y and vice versa. By assumption 4, $x(y)$ can attend meetings with y (x) to whom she appears acceptable at rate η_X (η_Y).

asset equation for $W(x|y)$ has the same form as before:

$$\begin{aligned} r_X W(x|y) &= \pi(x|y) - \delta\left[W(x|y) - U(x)\right] \\ \Leftrightarrow W(x|y) &= \frac{\pi(x|y) + \delta U(x)}{r_X + \delta} \\ \Leftrightarrow W(x|y) - U(x) &= \frac{\pi(x|y) - r_X U(x)}{r_X + \delta} \end{aligned} \tag{72}$$

which allows us to rewrite equation (71) as

$$\begin{aligned} r_X U(x) &= \eta_X \sum_{y\in M(x)} \left(\frac{\pi(x|y) - r_X U(x)}{r_X + \delta} - c_X\right) u_Y(y) \\ \Leftrightarrow r_X U(x) + \frac{\eta_X}{r_X+\delta} r_X U(x) \sum_{y\in M(x)} u_Y(y) &= \frac{\eta_X}{r_X+\delta} \sum_{y\in M(x)} \left(\pi(x|y) - c_X(r_X+\delta)\right) u_Y(y) \\ \Leftrightarrow r_X U(x) &= \frac{\sum_{y\in M(x)} \left(\pi(x|y) - c_X(r_X+\delta)\right) u_Y(y)}{\frac{r_X+\delta}{\eta_X} + \sum_{y\in M(x)} u_Y(y)} \end{aligned} \tag{73}$$

In words, the flow utility from search is determined by the expected share in a match with some $y \in M(x)$ conditional on $y \in M(x)$ in the first place. We can solve this out further if we determine $\pi(x|y)$, here in the split-the-surplus form:

$$\begin{aligned} \pi(x|y) &= r_X U(x) + q[f(x,y) - r_Y U(y) - r_X U(x)] \\ &= q[f(x,y) - r_Y U(y)] + (1-q) r_X U(x) \end{aligned}$$

Letting $\sum_{y\in M(x)} u_Y(y) \equiv \alpha$ to simplify the notation, equation (73) thus becomes

$$\begin{aligned} & r_X U(x) = \frac{q\sum_{y\in M(x)} [f(x,y) - r_Y U(y)]\, u_Y(y) + \alpha[(1-q) r_X U(x) - c_X(r_X+\delta)]}{\frac{r_X+\delta}{\eta_X} + \alpha} \\ \Leftrightarrow\quad & r_X U(x) \left[\frac{r_X+\delta}{\eta_X} + \alpha - \alpha(1-q)\right] = q \sum_{y\in M(x)} [f(x,y) - r_Y U(y)]\, u_Y(y) - \alpha c_X(r_X+\delta) \\ \Leftrightarrow\quad & r_X U(x) = \frac{q\sum_{y\in M(x)} [f(x,y) - r_Y U(y)]\, u_Y(y) - \alpha c_X(r_X+\delta)}{\frac{r_X+\delta}{\eta_X} + q\alpha} \end{aligned} \tag{74}$$

The last line specifies the reservation flow utility $r_X U(x)$ of agent x in terms of others' reservation flow utilities $r_Y U(y)$, which she has to take as given. We can now obtain the optimal stopping rule of agent x: she accepts a match whenever $W(x|y) - U(x) \geq 0$, and by equation (72) this means

$$\frac{\pi(x|y) - r_X U(x)}{r_X + \delta} \geq 0 \quad \Rightarrow \quad \pi(x|y) \geq r_X U(x)$$

We conclude that the optimal stopping rule is

$$\begin{aligned} S(x) &= \{y : W(x|y) \geq U(x)\} \\ &= \left\{ y : \pi(x|y) \geq \frac{q\sum_{y\in M(x)} [f(x,y) - r_Y U(y)]\, u_Y(y) - \alpha c_X(r_X+\delta)}{\frac{r_X+\delta}{\eta_X} + q\alpha} \right\} \end{aligned} \tag{75}$$

It remains to determine the matching set that we defined as

$$M(x) \equiv S(x) \cap \{y : x \in S(y)\}, \tag{76}$$

itself depending on $S(x)$. From the perspective of an individual agent who takes all others' strategies as given, the stopping rule and the definition of the matching set are two equations in the two unknowns $M(x)$ and $S(x)$.[22] As agents' choice of signal does not matter, it only remains to specify beliefs for the babbling equilibrium. Given that randomly drawn types are met, the only belief of an agent x consistent with actual equilibrium play is

$$\mu_X(y|\tilde{y}) = u_Y(y) \quad \forall y, \tilde{y} \in \Theta \tag{77}$$

and there are no off-equilibrium signals that would necessitate any more beliefs. The following proposition collects the results in this section:

Proposition 4 (Babbling equilibrium). *Any babbling equilibrium that exists will entail individual strategies like in equations (70) and (75), matching sets like in equation (76), and beliefs like in equation (77) for each $x \in \Theta$ and for each $y \in \Theta$, together with the pointwise steady state in equations (68) and (69).*

While we do not investigate existence of a babbling equilibrium in our model, note that a babbling equilibrium of our model with $r_X = r_Y \equiv r$, $\eta_X = \eta_Y \equiv \eta$, $q = 1 - q$, and $c_X = c_Y = 0$ is a special case of the equilibrium in Shimer and Smith (2000), whose existence they do prove.[23]

9.2 Pooling equilibrium

The only structural difference between the babbling equilibrium and a pooling equilibrium of our model is that signals have some role to play in every pooling equilibrium: each agent $x \in \Theta$ wants to meet only those agents $y \in \Theta$ who send the pooled equilibrium signal $\tilde{y}^P$. Likewise, only agents signalling $\tilde{x}^P$ will be accepted for a meeting with some $y \in \Theta$. This translates into the following meeting strategies for the pooling equilibrium:

$$D(x) = \{\tilde{y} : \tilde{y} = \tilde{y}^P\} \quad \forall x \in \Theta \quad \text{and} \quad D(y) = \{\tilde{x} : \tilde{x} = \tilde{x}^P\} \quad \forall y \in \Theta. \tag{78}$$

Since every $y \in \Theta$ indeed sends the signal $\tilde{y}^P$ in equilibrium, the strategy $D(x)$ prescribes that each agent $x \in \Theta$ seeks a meeting every time. By the same logic, each agent $y \in \Theta$ seeks a meeting every time, so that a meeting in fact results at every opportunity. This means that the type an agent actually faces in a meeting is a random draw from the other side of the market, just as in the babbling equilibrium. Assuming again that more frequent types will accordingly be met more frequently, the steady state in pooling equilibrium, the stopping rules as well as the matching sets are all as derived in the previous section. The only element of the babbling equilibrium to be adapted is agents' beliefs. Since all agents send the pooled signals in equilibrium, beliefs about the pooled signal will only be consistent with equilibrium play if

$$\mu_X(y|\tilde{y}^P) = u_Y(y) \quad \forall y \in \Theta \quad \text{and} \quad \mu_Y(x|\tilde{x}^P) = u_X(x) \quad \forall x \in \Theta \tag{79}$$

[22] Recall that $\pi(x|y)$ in the stopping rule is also given by the split-the-surplus formulation in terms of known variables.

[23] See the proof of proposition 1 in Shimer and Smith (2000), p. 351-353.

In addition, beliefs about off-equilibrium signals have to ensure that no agent has an incentive to meet anyone sending an off-equilibrium signal. Any pooling equilibrium will be of this form, leading to exactly the same expected payoffs as in the babbling equilibrium (apart from merely formal differences, e.g. $U(x) = V(\tilde{x}^P|\tilde{y}^P)$). As usual in signalling models, there is potentially a great number of pooling equilibria, one for every combination of pooled signals with according beliefs.

10 Relation to directed search models

10.1 Differences in frictions

Since directed search models feature signals in a matching context, it is more than legitimate to ask how our model differs and what it can contribute in relation to this particular literature. Directed search models commonly collect the interaction of employers and homogeneous job seekers on the labour market into three stages for each period. Firstly, all employers simultaneously choose wage offers that they advertise. Next, all job seekers observe the advertised wage offers and simultaneously choose which employer to apply to in this period. Finally, job seekers and employers match as follows. An employer who received exactly one application hires this one applicant. An employer who received several applications somehow selects, perhaps randomly, one of these applicants for the job. Employers who did not receive any applications can try again in the next period, as can job seekers who are not hired.

In principle, such a model could incorporate each kind of frictions: discounting, explicit search costs, and congestion. Congestion frictions clearly arise in directed search models because of a lack of co-ordination among job seekers, so that some employers receive several applications in a given period while others receive none, with the undesirable consequence that some job seekers remain unemployed in the current period alongside some unfilled vacancies. As to discounting and explicit search costs, however, it is highly problematic that virtually the entire directed search literature assumes commitment by employers to the wage offers they advertise, so that employers will not send a misleadingly attractive signal and then renege on it. Yet we explained in section 5.1 that the very existence of discounting or explicit search costs creates an incentive to mislead and then renege. Since the work of Diamond (1971) reneging has been known as a fundamental problem that can lead to complete market break-down even with infinitesimally small frictions of this kind. Hence, simply assuming that employers are committed to their advertised offers sidesteps these issues and typically renders discounting and explicit search costs irrelevant for agents' behaviour in the model. Accordingly, these two kinds of search frictions do not even appear anymore in recent contributions to this literature such as Burdett et al. (2001) and Shimer (2005) that focus entirely on congestion.

By contrast, our model incorporates discounting and explicit search costs and does not rule out reneging but leads to conditions for truthful signalling. In turn, congestion frictions are absent in our model. This is partly due simply to time being continuous in our model. In directed search models, the stages are effectively the same as discrete time periods in which agents move simultaneously. With continuous time, however, simultaneous moves happen with zero probability. Hence, in a model with homogeneous job seekers, one job seeker would always be the first applicant, and the employer cannot gain from waiting for more applicants. Then congestion never arises.

Of course, congestion frictions do exist in the real world despite continuous time, as many employers (though not all) accept applications until some specified deadline and interview only after the deadline. If the employer instead met the first applicant with a suitable signal, this would always save her some costs (at least the expected costs of processing further applications and of losing the first suitable applicant to another employer in the meanwhile). To wait until the deadline nevertheless must therefore carry some benefit for the employer. There may be an option value of waiting because the employer is uncertain what types of applicants might be attracted by the job offer, i.e. the first suitable applicant might still be less suitable than later applicants. In particular, it may take some time for word of the job offer to reach the most suitable applicants. Another explanation could be that some signals are imprecise or deliberately misleading, prompting employers to interview a selection of applicants and compare their interview performance. Yet in the separating equilibrium of our model, none of these benefits exist: agents know with whom they can match, communication is instant, and signals are both precise and truthful. In our model, employers therefore do meet and hire the first applicant with a (uniquely) suitable signal, thereby preventing any congestion.

10.2 Exogenous arrival rates

Following the work of Peters (1991), it has become a central theme of directed search models that employers seek to influence the arrival rate they face by their choice of wage offer. Such endogenous arrival rates are then the higher the more competitive is the wage offer. In a standard model with homogenous job seekers, this implies a trade-off for job seekers between a high wage and a high probability of obtaining the advertised job: a comparatively high wage attracts many applicants, only one of which will be hired. But this appears to be largely driven by the homogeneity of job seekers: with heterogeneous job seekers, employers would select the applicant with the highest type. For this applicant, it is irrelevant how many job seekers with lower types apply.[24] Because this applicant is, however, still more likely to face competition from equally high types for a high-wage job than for a low-wage job, a (much reduced) trade-off may remain with heterogeneous job seekers.[25]

The heterogeneous agents in our model signal their types truthfully in the separating equilibrium, so that some of them can be selected for meetings on the basis of their signals. Hence those who would be selected do not fear the competition of lower types. Moreover, as we explained in the preceding section, it does not happen in our model that two equal types apply simultaneously, so that not even the reduced trade-off remains. There remains the possibility that certain lower types are somehow easier to meet because there are more of them around. Assumption 1 formally rules out different frequencies of types, but we motivated it only as the assumption of a smooth type distribution in steady state, resulting from a logic of arbitrage in type changes. With a smooth type distribution, the gains from higher meeting rates with lower types are unlikely to outweigh the utility loss over the entire match duration. To formalise this point, consider an agent $y \in \Theta$ who,

[24] As a real-world example, the congestion for jobs at certain top consultancies does not discourage truly qualified applicants: recruiters very quickly identify outstanding applications and the congestion thus does not have a significant impact on these applicants' meeting rates.

[25] The findings of Inderst (2005) even suggest that the trade-off disappears completely in an equilibrium à la Moen (1997) with heterogeneous job seekers: in his model, the high type obtains a higher wage and matches faster than the low type.

for the next meeting only, has the choice between meeting a high type x_H at rate η_1 or meeting a lower type x_L at rate η_2, where $\eta_2 > \eta_1$. In expectation, a meeting with x_L will then occur a time $\frac{1}{\eta_1} - \frac{1}{\eta_2}$ earlier than a meeting with x_H. We suppose that the meeting leads to a match in either case, we discount to the beginning of the match with x_L, and we ignore that a match with x_H is expected to terminate later in time because any utility at that time is heavily discounted. The higher meeting rate will make agent y opt for meeting x_L if

$$\begin{aligned} \int_0^{\frac{1}{\delta}} \pi(x_L|y)e^{-r_Y t}dt &> \int_{\frac{1}{\eta_1}-\frac{1}{\eta_2}}^{\frac{1}{\delta}} \pi(x_H|y)e^{-r_Y t}dt \\ \Leftrightarrow \int_0^{\frac{1}{\eta_1}-\frac{1}{\eta_2}} \pi(x_L|y)e^{-r_Y t}dt &> \int_{\frac{1}{\eta_1}-\frac{1}{\eta_2}}^{\frac{1}{\delta}} [\pi(x_H|y) - \pi(x_L|y)]e^{-r_Y t}dt \end{aligned}$$

When x_L is close to x_H in this condition, a smooth type distribution implies that meeting rates are also close. A small $\frac{1}{\eta_1} - \frac{1}{\eta_2}$ then allows the condition to fail even when $\pi(x_H|y) - \pi(x_L|y)$ is small. In turn, when the former is large, the condition can still fail because the latter is then also large. Therefore, a smooth type distribution tends to imply that differences in type frequencies do not affect agents' behaviour. Then one can just as well assume exogenous and identical meeting rates, as we did in assumption 4.

11 Conclusions

This paper has introduced costless signals into a setting of two-sided search with transferable utility and search frictions. We find a unique separating equilibrium characterised by perfect positive assortative matching, minimised incidence of search costs, and maximised overall match output. The non-separating equilibria of our model can be identified with the equilibrium in Shimer and Smith (2000), abstracting from elements that are specific to models with signals. Given strong incentives for effective communication to facilitate search and given the pervasive use of such communication in real-world matching markets, the unique separating equilibrium appears as the only practically relevant equilibrium of our model. Positive assortative matching in this equilibrium only relies on a sufficient degree of supermodularity of the match production function. This implies that our model proposes a solution to the paradox in Shimer and Smith (2000), where positive assortative matching requires not only supermodularity as such, but also log-supermodularity of the first derivative and log-supermodularity of the cross-partial derivative. In fact, the comparatively weak condition on supermodularity in our model does not merely ensure positive assortative matching despite discounting, but perfect positive assortative matching despite discounting and explicit search costs.

We are aware of only one other solution to this paradox, the model by Chade (2006), where a form of stochastic positive assortative matching arises without any condition on supermodularity. However, these results rely on exogenous yet informative signals. We believe that real-world agents can strategically influence their signals, so that the issue of deliberately misleading signals cannot be ignored. Our paper has gone to great length to show that the same condition on supermodularity ensures both truthful signalling and perfect positive assortative matching. While assortative matching in real-world markets is typically imperfect, this is not necessarily due to signals being beyond agents' control. To name but three potential explanations among many more, actual types might be multidimensional or otherwise much less clear-cut than simple scalars, agents might be imperfectly informed about their own type, or agents' personal networks might undermine meritocratic sorting. Our results suggest that, if appropriate policies made real-world matching markets more similar to the simplified world of our model, the efficiency gains could be large.

Future work might introduce a long side to the matching market, so that there are excess types, and examine how such a situation might converge to an equilibrium. A comparative statics analysis appears worthwhile, in particular with regards to the induced changes in efficiency. More generally, we would argue that assortative matching becomes more important as technological and societal progress favours specialisation. At the same time, new means have appeared of effective and rapid communication that might, as in our paper, support assortative matching. The combination of these two developments offers ample scope for further research.

References

[1] Adachi, Hiroyuki (2003): "A Search Model of Two-sided Matching under Non-transferable Utility," *Journal of Economic Theory* 113, 182-198

[2] Atakan, Alp E. (2006): "Assortative Matching with Explicit Search Costs," *Econometrica* 74, 667-680

[3] Becker, Gary S. (1973): "A Theory of Marriage: Part I," *Journal of Political Economy* 81, 813-846

[4] Burdett, Kenneth and Coles, Melvyn G. (1999): "Long-Term Partnership Formation: Marriage and Employment," *Economic Journal* 109, F307-F334

[5] Burdett, Kenneth, Shi, Shouyong and Wright, Randall (2001): "Pricing and Matching with Frictions," *Journal of Political Economy* 109, 1060-1085

[6] Chade, Hector (2006): "Matching with Noise and the Acceptance Curse," *Journal of Economic Theory* 129, 81-113

[7] Coles, Peter and Niederle, Muriel (2007): "Signaling in Matching Markets," Stanford University mimeo

[8] Crawford, Vincent P. and Sobel, Joel (1982): "Strategic Information Transmission," *Econometrica* 50, 1431-1451

[9] Diamond, Peter A. (1971): "A Model of Price Adjustment," *Journal of Economic Theory* 3, 156-168

[10] Eeckhout, Jan and Kircher, Philipp (2008): "Sorting and Decentralized Price Competition," University of Pennsylvania mimeo

[11] Hopkins, Ed (2007): "Job Market Signaling of Relative Position, or Becker Married to Spence," University of Edinburgh mimeo

[12] Hoppe, Heidrun C., Moldovanu, Benny, and Sela, Aner (2008): "The Theory of Assortative Matching Based on Costly Signals," *Review of Economic Studies* (forthcoming)

[13] Inderst, Roman (2005): "Competitive Search Markets with Heterogeneous Workers," *European Economic Review* 49, 1525-1542

[14] Mare, Robert D. (1991): "Five Decades of Educational Assortative Mating," *American Sociological Review* 56, 15-32

[15] Moen, Espen R. (1997): "Competitive Search Equilibrium," *Journal of Political Economy* 105, 385-411

[16] Morgan, Peter B. (1998): "A Model of Search, Coordination, and Market Segmentation," University at Buffalo mimeo

[17] Peters, Michael (1991): "Ex Ante Price Offers in Matching Games Non-Steady States," *Econometrica* 59, 1425-1454

[18] Shimer, Robert (2005): "The Assignment of Workers to Jobs in an Economy with Coordination Frictions," *Journal of Political Economy* 113, 996-1025

[19] Shimer, Robert and Smith, Lones (2000): "Assortative Matching and Search," *Econometrica* 68, 343-369

[20] Smith, Lones (2006): "The Marriage Model with Search Frictions," *Journal of Political Economy* 114, 1124-1146

[21] Spence, Michael (1973): "Job Market Signaling," *Quarterly Journal of Economics* 87, 355-374

[22] Telser, Lester G. (1978): *Economic Theory and the Core,* Chicago: University of Chicago Press

[23] Weitzman, Martin L. (1979): "Optimal Search for the Best Alternative," *Econometrica* 47, 641-654